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PERMUTATION

1.1 Introduction

In many problems, we want to find the number of ways a set of objects can be arranged under different restrictions.

If a certain number of objects are given, each of the different arrangements that can be made out of them by taking some or all of them at a time is known as permutation.

Example: Let us consider three letters a, b, and c. Taking all three at a time, we have six different arrangements as

a b c
a c b
b a c
b c a
c a b
c b a

Each arrangement above is called a permutation of three letters.

Further, if we take two letter out of these three letters, the possible arrangements are :

a b
a c
b a
b c
c a
c b

These arrangements are also six in numbers. So, the number of distinct arrangements of three letters taking two at a time is six.

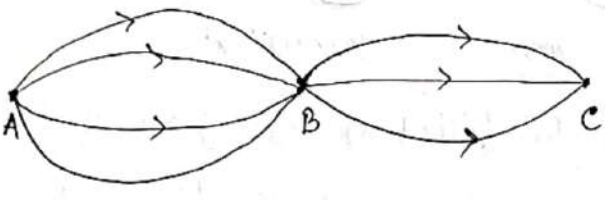
1.2 Fundmental rules of counting

1. If one operation can be performed in m different ways and another operation can be performed in n different ways and both can not be performed at the same time Then the total number of ways in which either of the two operations can be performed is $(m+n)$

Example: A woman has decided to shop at one store today, either in the north part of town or in the south part of town. If she visits the north part of town, she will shop at either a mall, a furniture store or a jewellery store. If she visits the south part of the town then she will shop at either a cloth store or a shoe store. So, there are $3+2=5$ possible shops the woman could end up shopping at today.

2. If one operation can be performed in m different ways and another operation can be performed in n different ways, then the total number of ways in which both the operations can be performed simultaneously or successively is $m \times n$.

Example: Let A, B and C be three plays. There are 4 different roads connecting A and B. Also, ~~there~~ there are 3 different roads connecting B to C



So, ~~there~~ there are $4 \times 3 = 12$ ways of going from A to C through B by road, because for each road from A to B, you will get 3 options to go from B to C. So, for 4 roads from A to B, you get 4×3 ways to go from A to C through B by ~~B~~ road.

<u>Factorial Notation</u>: The product of first n natural numbers is called factorial of n or factorial n and denoted by $\underline{|n}$ or $n!$. That is $n! = n(n-1)(n-2)\cdots 3.2.1$

For example, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

We define $0! = 1$

Notes: 1. Factorials of numbers other than natural numbers and 0 are not defined.

2. For any natural number n, $n! = n \times (n-1)!$

1.3 Results on permutation

Result 1.3.1 The number of permutations of n distinct objects taken $r\ (\leq n)$ at a time, denoted by ${}^nP_r$, is given by

$$ {}^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!} $$

Proof: The number of permutation of n distinct objects taken $r\ (\leq n)$ at a time is equivalent to the number of ways of filling up r places by r objects taken from n distinct objects.

1	2	3	. . .	r
n ways	$(n-1)$ ways	$(n-2)$ ways		$(n-r+1)$ ways

The first place can be filled up by any one of the n objects in n different ways. After filling up the first place, the second place can be filled up by any one of the remaining $(n-1)$ objects in $(n-1)$ different ways. Again, the third place can be filled up by any one of the remaining $(n-2)$ objects in $(n-2)$ different ways. Proceeding similarly, the rth place can be filled by any of the remaining $\{n-(r-1)\}$ in $(n-r+1)$ different ways. Finally, applying fundamental rules of counting the number of ways of filling up $r\ (\leq n)$ places is $n(n-1)(n-2)\cdots(n-r+1)$. So, the number of permutation

$$= {}^nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

Again, $n(n-1)(n-2)\cdots(n-r+1) = n(n-1)(n-2)\cdots(n-r+1)\dfrac{(n-r)!}{(n-r)!}$

$$= \frac{n!}{(n-r)!}$$

Hence, ${}^nP_r = \dfrac{n!}{(n-r)!}$

Notes: 1. ${}^nP_0 = 1$ and ${}^nP_1 = n$

2. The number of ~~of~~ permutations of n objects taken all at a time is

$${}^nP_n = n!$$

Result 1.3.2 The number of permutations of n objects taken all together, the objects are not all different, is given by

$\dfrac{n!}{p!\,q!\,r!}$, where p objects are alike of first kind, q objects are alike of second kind and r objects are alike of third ~~f~~ kind.

Proof: Let us denote n objects by n symbols of which there p symbols are 'A', q symbols are 'B' and r symbols are 'C' and the rest of the symbols are different. Suppose the required number of permutation be x. In each of the ~~x~~ permutation, there are p alike symbols 'A', q alike symbols 'B' and r alike symbols 'C' and the rest $(n-p-q-r)$ symbols are different. The total number of symbols in each of the permutation is n. Consider any one of the x permutations and replace p number of A's by p distinct symbols $A_1, A_2, \ldots, A_p$. The p distinct