3.2 Binomial Theorem

Statement of Binomial Theorem for positive integral index

For any positive integer n,

$$(a+x)^n = a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \cdots + {}^nC_r a^{n-r}x^r + \cdots + x^n$$

Notes 1. Since ${}^nC_0 = 1 = {}^nC_n$, the binomial theorem can be written as $(a+x)^n = {}^nC_0 a^n x^0 + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \cdots + {}^nC_r a^{n-r}x^r + \cdots + {}^nC_n x^n$

2. Since ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ appears as the coefficients of the respective terms in the expansion of $(a+x)^n$, they are known as binomial coefficients in the expansion of $(a+x)^n$.

3. The number of terms in the binomial expansion of $(a+x)^n$ is $(n+1)$ and the sum of the powers of a and x in each term is n.

4. Since ${}^nC_r = {}^nC_{n-r}$, the binomial ~~coefficients~~ coefficients of the terms equidistant from the beginning and the end are equal.

5. If we substitute $a=1$ in the binomial theorem then

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r + \cdots + x^n$$

6. If we replace x by $(-x)$ in the binomial theorem, then

$$(a-x)^n = a^n - {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 - \cdots + (-1)^r\, {}^nC_r a^{n-r}x^r + \cdots + (-1)^n x^n$$

7. If we replace x by $(-x)$ in formula of 5., we have

$$(1-x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - \cdots + (-1)^r\, {}^nC_r x^r + \cdots + (-1)^n x^n$$

8. If we put $x=1$, in formula of 5, we have

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_r + \cdots + {}^nC_n$$

9. If we put $x=-1$ in formula of 7, we have

$$0 = (1-1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots + (-1)^n\, {}^nC_n$$

Thus $^nc_0 + {}^nc_2 + {}^nc_4 + \cdots = {}^nc_1 + {}^nc_3 + {}^nc_5 + \cdots$

10. From formula of 8,

$2^n = {}^nc_0 + {}^nc_1 + {}^nc_2 + \cdots + {}^nc_n$. Again from formula of 9,

we have $^nc_0 + {}^nc_2 + {}^nc_4 + \cdots = {}^nc_1 + {}^nc_3 + {}^nc_5 + \cdots = k$ (say)

So, $2k = 2^n$ or, $k = 2^{n-1}$

Hence, $^nc_0 + {}^nc_2 + {}^nc_4 + \cdots = {}^nc_1 + {}^nc_3 + {}^nc_5 + \cdots = 2^{n-1}$

3.3 General term of $(a+x)^n$

Let t_r be the rth term of the binomial expansion of $(a+x)^n$, then we see that

$$t_1 = t_{0+1} = {}^nc_0 \, a^{n-0} x^0$$
$$t_2 = t_{1+1} = {}^nc_1 \, a^{n-1} x^1$$
$$t_3 = t_{2+1} = {}^nc_2 \, a^{n-2} x^2$$

So, generalizing this, we get,

$$t_{r+1} = {}^nc_r \, a^{n-r} x^r$$

The $(r+1)$th term $t_{r+1} = {}^nc_r \, a^{n-r} x^r$, $r = 0, 1, 2, \ldots, n$ is called the general term of the binomial expansion of $(a+x)^n$.

Notes 1. General term of $(a-x)^n$ is

$$t_{r+1} = (-1)^r \, {}^nc_r \, a^{n-r} x^r$$

2. General term of $(1+x)^n$ is

$$t_{r+1} = {}^nc_r \, x^r$$

3. General term of $(1-x)^n$ is

$$t_{r+1} = (-1)^r \, {}^nc_r \, x^r$$

3.4 Middle term(s) of $(a+x)^n$

The binomial expansion of $(a+x)^n$ contains $(n+1)$ terms. If n be even then the number of terms is odd. If n be odd then the number of term is even. So, when n is even, we have only one middle term which is the term $t_{\frac{n}{2}+1}$ in the binomial expansion of $(a+x)^n$.

~~The value of the middle term of $\frac{n}{2}$ are~~

So, here the middle term is $t_{\frac{n}{2}+1} = {}^{n}c_{n/2}\, a^{n/2} x^{n/2}$.

Again for odd n, there are two middle terms and they are $t_{\frac{n+1}{2}}$ and $t_{\frac{n+1}{2}+1}$ in the binomial expansion of $(a+x)^n$.

Hence, the two middle terms are,

$$\cancel{t_{\frac{n+1}{2}} = {}^{n}c_{\frac{n+1}{2}}\, a \; x^{\frac{n+1}{2}}}$$

$$\cancel{= {}^{n}c_{\frac{n-1}{2}}\, a^{\frac{n-1}{2}}\, x}$$

$$\text{and } \cancel{t_{\frac{n+1}{2}+1} = {}^{n}c_{\frac{n+1}{2}}\, a^{\frac{n-1}{2}}\, x^{\frac{n+1}{2}}}$$

$$t_{\frac{n+1}{2}} = t_{\frac{n-1}{2}+1} = {}^{n}c_{\frac{n-1}{2}}\, a^{\frac{n+1}{2}}\, x^{\frac{n-1}{2}}$$

$$\text{and } t_{\frac{n+1}{2}+1} = {}^{n}c_{\frac{n+1}{2}}\, a^{\frac{n-1}{2}}\, x^{\frac{n+1}{2}}$$

3.5 Equidistant terms and Coefficients

The binomial expansion of $(a+x)^n$ contains $(n+1)$ terms. The rth term from the beginning and the rth term from the end are called equidistant terms, for $r=1,2,\ldots,n+1$. Thus the first term t_1 is and last term t_{n+1} are equidistant terms; Similarly, t_2 and t_n are equidistant terms and so on. Thus in general, t_r and $t_{(n+2-r)}$ are equidistant terms, $r=1,2,\ldots,n+1$

Now $t_r = {}^nc_{r-1}\, a^{n-r+1}\, x^{r-1}$ and $t_{(n+2-r)} = t_{n+1-r+1} = {}^nc_{n+1-r}\, a^{r-1}\, x^{n+1-r}$

Again, ${}^nc_{r-1} = {}^nc_{n+1-r}$. Thus, the binomial coefficients of equidistant terms are equal.

3.6 Greatest Binomial Coefficient(s)

The greatest binomial coefficient in the expansion of $(a+x)^n$ is

${}^nc_{n/2}$, if n is even

and ${}^nc_{\frac{n-1}{2}}$ and ${}^nc_{\frac{n+1}{2}}$, if ~~n is~~ n is odd

~~Here~~ Here ${}^nc_{\frac{n-1}{2}} = {}^nc_{\frac{n+1}{2}}$

3.7 Properties of ~~Binom~~ Binomial Coefficients

The binomial coefficients have the following properties:

1. ${}^nc_0 + {}^nc_1 + {}^nc_2 + \cdots + {}^nc_n = 2^n$
2. ${}^nc_0 - {}^nc_1 + {}^nc_2 - \cdots + (-1)^n\, {}^nc_n = 0$
3. ${}^nc_0 + {}^nc_2 + {}^nc_4 + \cdots = {}^nc_1 + {}^nc_3 + {}^nc_5 + \cdots = 2^{n-1}$