4. In the expansion of $(a+x)^n$, the binomial coefficient of terms equidistant from the beginning and end are equal. That is, the coefficient of $(r+1)$th term from the beginning = the ~~coef~~ coefficient of $(r+1)$th term from the end.

Example 1 Write down the expansion of the following binomial expression $\left(2x-\frac{1}{x}\right)^5$

Solution: By using Binomial Theorem,

$$\left(2x-\frac{1}{x}\right)^5 = (2x)^5 - {}^5C_1\,(2x)^4\left(\frac{1}{x}\right) + {}^5C_2\,(2x)^3\left(\frac{1}{x}\right)^2 - {}^5C_3\,(2x)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\,(2x)\left(\frac{1}{x}\right)^4 - \left(\frac{1}{x}\right)^5$$

$$= 32x^5 - 5\left(16x^4\right)\left(\frac{1}{x}\right) + 10\left(8x^3\right)\left(\frac{1}{x^2}\right) - 10\left(4x^2\right)\left(\frac{1}{x^3}\right) + 5(2x)\left(\frac{1}{x^4}\right) - \frac{1}{x^5}$$

$$= 32x^5 - 80x^3 + 80x - \frac{40}{x} + \frac{10}{x^3} - \frac{1}{x^5}$$

Example 2 Using binomial theorem, find the value of $(99)^4$

Solution: $(99)^4 = (100-1)^4$

$$= (100)^4 - {}^4C_1(100)^3 + {}^4C_2(100)^2 - {}^4C_3(100) + 1$$

$$= 100000000 - 4000000 + 60000 - 400 + 1$$

$$= 96059601$$

Example 3 Evaluate $(\sqrt{3}+1)^6 + (\sqrt{3}-1)^6$

Solution: $(\sqrt{3}+1)^6 = (\sqrt{3})^6 + {}^6C_1(\sqrt{3})^5 + {}^6C_2(\sqrt{3})^4 + {}^6C_3(\sqrt{3})^3 + {}^6C_4(\sqrt{3})^2 + {}^6C_5(\sqrt{3}) + 1$

$$= 27 + 6(9\sqrt{3}) + 15(9) + 20(3\sqrt{3}) + 15(3) + 6\sqrt{3} + 1$$

$$= 27 + 54\sqrt{3} + 135 + 60\sqrt{3} + 45 + 6\sqrt{3} + 1$$

Similarly, $(\sqrt{3}-1)^6 = 27 - 54\sqrt{3} + 135 - 60\sqrt{3} + 45 - 6\sqrt{3} + 1$

So, $(\sqrt{3}+1)^6 + (\sqrt{3}-1)^6 = 2(27+135+45+1) = 2 \times 208 = 416$

Example 4 Find the coefficient of x^2 in the expansion of $\left(\frac{x}{2} + \frac{2}{x}\right)^8$.

Solution: The $(r+1)$th term in the expansion of $\left(\frac{x}{2} + \frac{2}{x}\right)^8$

is $t_{r+1} = {}^8c_r \left(\frac{x}{2}\right)^{8-r} \left(\frac{2}{x}\right)^r = {}^8c_r \, 2^{2r-8} \, x^{8-2r}$

If t_{r+1} be the term containing x^2, then

$$x^{8-2r} = x^2$$

or, $8 - 2r = 2$

or, $2r = 6$

or, $r = 3$

So, the coefficient of x^2 in the expansion of $\left(\frac{x}{2} + \frac{2}{x}\right)^8$

is ${}^8c_3 \, 2^{2\times3-8} = {}^8c_3 \, 2^{-2} = \frac{8\times7\times6}{3\times2\times1} \times \frac{1}{2^2} = 14$

Example 5 Find the coefficient of x^{11} in the expansion of $(1-2x+3x^2)(1+x)^{11}$

Solution: Using binomial theorem, the given expression can be written as

$(1-2x+3x^2)\left(1 + {}^{11}c_1 x + {}^{11}c_2 x^2 + \dots + {}^{11}c_9 x^9 + {}^{11}c_{10} x^{10} + 1x^{11}\right)$

So, the coefficient of x^{11} in the above expression is

$= 1\times 1 - 2\times {}^{11}C_{10} + 3\times {}^{11}C_{9}$

$= 1 - 2\times {}^{11}C_{1} + 3\times {}^{11}C_{2} = 1 - 2\times 11 + 3\times 55 = 1-22+165$

$= 144$

Example 6 Find the term independent of x in the expansion of $\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9$

Solution: Let the $(r+1)$th term t_{r+1} be the term independent of x in the expansion of $\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9$

Now, $t_{r+1} = {}^{9}C_{r}\left(\frac{4}{3}x^2\right)^{9-r}\left(-\frac{3}{2x}\right)^{r}$

$= {}^{9}C_{r}\left(\frac{4}{3}\right)^{9-r}\left(-\frac{3}{2}\right)^{r}\left(x^2\right)^{9-r}\left(\frac{1}{x}\right)^{r}$

$= {}^{9}C_{r}\left(\frac{4}{3}\right)^{9-r}\left(-\frac{3}{2}\right)^{r} x^{18-3r}$

It is given that the term is independent of x,

So, $18 - 3r = 0$

or $r = 6$

So, the 7th term in the expansion of $\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9$ is independent of x and the term is

$$t_7 = {}^{9}C_{6}\left(\frac{4}{3}\right)^{9-6}\left(-\frac{3}{2}\right)^{6} = 2268$$

Example 7 Find the middle term in the expansion of $\left(2x - \frac{1}{3x}\right)^{10}$

[illegible] terms in the expansion of [illegible] Hence there exists only [illegible] $\left(\frac{10}{2}+1\right)$th term or 6 th term

$= [illegible]\, (3x)^{10-5}\left(-\frac{1}{3x}\right)^5$

$= [illegible] = {}^{10}C_5\, 2^5\left(-\frac{1}{3}\right)^5$

$= [illegible] = 252\times\left(\frac{2}{3}\right)^5$

[illegible] using Binomial theorem prove that

$3^{2n+2} - 8n - 9$ is divisible by 64.

Solution. The given expression,

$3^{2n+2} - 8n - 9$

$= (3^2)^{n+1} - 8n - 9 = 9^{n+1} - 8n - 9 = (1+8)^{n+1} - 8n - 9$

$= [illegible] \times 8^2 + \cdots + 8^{n+1}\} - (n+1)8 - 1$

$= [illegible] \times 8^3 + \cdots + 8^{n+1}\}$

$= [illegible] \times 8 + \cdots + 8^{n-1})$

[illegible] the expression within the bracket is a positive integer, [illegible] expression is divisible by 64.

[illegible] of $(r+3)$th term in the expansion [illegible] the coefficients of $(3r+2)$th term, [illegible]