Solution: We know that the $(r+3)$th in the expansion of $(1+x)^{47}$ is

$$t_{r+3} = {}^{47}C_{r+2}\, x^{r+2}$$

Again, the $(3r+2)$th term in the expansion of of $(1+x)^{47}$ is

$$t_{3r+2} = {}^{47}C_{3r+1}\, x^{3r+1}$$

It is given that coefficient of t_{r+1} is equal to the coefficient of t_{3r+2}. Thus, ${}^{47}C_{r+2} = {}^{47}C_{3r+1}$

This is possible if either $r+2 = 3r+1$ or $r+2 = 47-(3r+1)$

That is, either $r = \frac{1}{2}$ or $r = 11$

But $r = \frac{1}{2}$ is not possible, because r is an integer.

So, $r = 11$. Therefore, the two terms are

$$t_{r+3} = t_{14} = {}^{47}C_{11}\, x^{13}$$

and $$t_{3r+2} = t_{35} = {}^{47}C_{34}\, x^{34}$$

Example 10 Prove that

(a) $^nC_1 + 2\,^nC_2 + 3\,^nC_3 + \cdots + n\,^nC_n = n.2^{n-1}$

(b) $\left(^nC_0 + {}^nC_1\right)\left(^nC_1 + {}^nC_2\right)\left(^nC_2 + {}^nC_3\right)\cdots\left(^nC_{n-1} + {}^nC_n\right) = {}^nC_1 \cdot {}^nC_2 \cdots {}^nC_n \dfrac{(n+1)^n}{n!}$

Solution: $^nC_1 + 2\,^nC_2 + 3\,^nC_3 + \cdots + n\,^nC_n$

$$= n + 2.\frac{n(n-1)}{2!} + 3\,\frac{n(n-1)(n-2)}{3!} + \cdots + n$$

$$= n\left[1 + {}^{n-1}c_1 + {}^{n-1}c_2 + \cdots + {}^{n-1}c_{n-1}\right] = n(1+1)^{n-1} = n.2^{n-1}$$

(b) $\left(\dfrac{{}^nc_0 + {}^nc_1}{{}^nc_1}\right)\left(\dfrac{{}^nc_1 + {}^nc_2}{{}^nc_2}\right)\left(\dfrac{{}^nc_2 + {}^nc_3}{{}^nc_3}\right)\cdots\left(\dfrac{{}^nc_{n-1} + {}^nc_n}{{}^nc_n}\right)$

$$= \frac{{}^{n+1}c_1}{{}^nc_1}\cdot\frac{{}^{n+1}c_2}{{}^nc_2}\;\frac{{}^{n+1}c_3}{{}^nc_3}\cdots\;\frac{{}^{n+1}c_n}{{}^nc_n}$$

$$= \left(\frac{n+1}{n}\right)\left(\frac{n+1}{n-1}\right)\left(\frac{n+1}{n-2}\right)\cdots\left(\frac{n+1}{1}\right) = \frac{(n+1)^n}{n(n-1)(n-2)\cdots 1} = \frac{(n+1)^n}{n!}$$

Example 11 In the expansion of $(x+1)^{10}$, find (i) the total number of terms and (ii) the general term.

Solution: (i) We know that the total number of terms in the expansion of $(x+1)^n$ is $n+1$. So, the total number of terms in $(x+1)^{10}$ is 11.

ii) The general term of the expansion of $(x+1)^{10}$ is the $(r+1)$th term t_{r+1} given by

$$t_{r+1} = {}^{10}c_r\, x^{10-r}(1)^r = {}^{10}c_r\, x^{10-r}$$

Example 12 If the number of terms in $(1+x)^n$ be 11, find the 5th term and the value of n.

Solution: The total number of term in the expansion of $(1+x)^n$ is $n+1$. So, $n+1 = 11$

$$\text{or,}\quad n = 10$$

The 5 th term in the expansion of $(1+x)^{10}$ is

$$t_5 = {}^{10}c_4 (1)^{10-4} x^4 = {}^{10}c_4\, x^4$$

$$= \frac{10\times9\times8\times7}{4\times3\times2} x^4 = 210x^4$$

Example 13 Find the term independent of x in $\left(2x - \frac{1}{\sqrt{x}}\right)^{15}$.

Solution: Let the (r+1)th term t_{r+1} be independent of x in the expansion of $\left(2x - \frac{1}{\sqrt{x}}\right)^{15}$.

Now, $$t_{r+1} = {}^{15}c_r (2x)^{15-r}\left(-\frac{1}{\sqrt{x}}\right)^r$$

$$= {}^{15}c_r\, 2^{15-r} x^{15-r} (-1)^r \left(\frac{1}{\sqrt{x}}\right)^r$$

$$= {}^{15}c_r\, 2^{15-r} (-1)^r x^{15-\frac{3r}{2}}$$

So, $$15 - \frac{3r}{2} = 0$$

or, $$r = 10$$

Thus, $(10+1) = 11$ th term is independent of x and the term is $t_{11} = {}^{15}c_{10}\, 2^{15-10} (-1)^{10}$

$$= {}^{15}c_{10} \cdot 2^5 = 96096$$

Example 14 If the coefficients of the 5th, 6th and 7th term in the expansion of $(1+x)^n$ are in AP (arithmetic progression)

Solution : The $(r+1)$th term in the expansion of $(1+x)^n$ is

$t_{r+1} = {}^nc_r x^r$. Now,

$$t_5 = {}^nc_4 x^4$$
$$t_6 = {}^nc_5 x^5$$
$$t_7 = {}^nc_6 x^6$$

It is given that the coefficients are in AP; that is,

$$^nc_5 - {}^nc_4 = {}^nc_6 - {}^nc_5$$

or, $$^nc_6 + {}^nc_4 = 2.{}^nc_5$$

or, $$\frac{n!}{(n-6)!\,6!} + \frac{n!}{(n-4)!\,4!} = 2.\frac{n!}{(n-5)!\,5!}$$

or, $$\frac{1}{6\times 5} + \frac{1}{(n-4)(n-5)} = \frac{2}{5(n-5)}$$

or, $$(n-4)(n-5) + 30 = 12(n-4)$$

or, $$n^2-9n+20+30 = 12n-48$$

or, $$n^2-21n+98 = 0$$

or, $$(n-7)(n-14) = 0 \quad \text{So, } n = 7 \text{ or } 14$$

Example 15 Obtain the 5th term in the expansion of $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$