Solution: The (r+1)th term in the expansion of $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$

$$t_{r+1} = {}^{10}C_r \left(\frac{x}{a}\right)^{10-r} \cdot \left(\frac{a}{x}\right)^{r}$$

Thus $t_5 = {}^{10}C_4 \left(\frac{x}{a}\right)^{10-4} \cdot \left(\frac{a}{x}\right)^{4} = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}\left(\frac{x}{a}\right)^{6} \cdot \left(\frac{a}{x}\right)^{4} = 210\left(\frac{x}{a}\right)^{2}$

Example 16

Find the term independent of the variable x in the expansion of $\left(x+\frac{1}{x}\right)^{10}$

Solution: Let the (r+1)th term, i.e., t_{r+1}, be independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{10}$

Now, $t_{r+1} = {}^{10}C_r \, x^{10-r} \cdot \left(\frac{1}{x}\right)^{r} = {}^{10}C_r \, x^{10-2r}$

From the given condition, $10-2r=0$

or $r=5$

So, (r+1)th = (5+1)th = 6th term is independent of x

and $t_6 = {}^{10}C_5 = \frac{10!}{5!\,5!} = 252$

Exercises 1. Find the middle term of $(a+x)^{10}$

2. Write down the fifth term in the expansion of $\left(x+\frac{1}{x}\right)^{8}$

3. Find the value of term independent of

x in the expansion of $\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9$

4. Use binomial theorem to find the value of $(99)^4$.

5. Find the coefficient of x^4 in the expansion of $\left(x^4 + \frac{1}{x^2}\right)^{15}$.

6. Find the coefficient of x^7 in the expansion of $\left(2x^2 + \frac{1}{4x}\right)^{11}$.

7. Find the term of independent of x in the expansion of $\left(x - \frac{2}{x^2}\right)^{15}$

8. Expand by binomial theorem $\left(\frac{x}{2} + \frac{3}{x}\right)^7$

9. Evaluate $(\sqrt{5} + \sqrt{2})^5 + (\sqrt{5} - \sqrt{2})^5$

10. Find the coefficient of x^{11} in the expansion of $(1 - 2x + 3x^2)(1 + x)^{11}$

11. Find the coefficient of a^{11} in the expansion of $\left(5a^3 - \frac{2}{a^2}\right)^{13}$.

12. Find the middle term(s) in the expansion of

(i) $\left(x + \frac{2}{x}\right)^9$ (ii) $\left(x - \frac{1}{x}\right)^{12}$

13. In the expansion of $(1+x)^{25}$, the coefficient of $(2r+1)$th term and $(r+5)$th terms are equal. Find the value of r.

14. Using binomial theorem, prove that

a $14^n - 13n - 1$ is divisible by 169

15. Prove that

$$^nc_0 + 2.^nc_1 + 3.^nc_2 + \cdots + (n+1)\,^nc_n = (n+2)\cdot 2^{n-1}$$

LOGARITHM

5.1. Definition of Logarithm

Let a and M be two real numbers such that $a \neq 1$, $a > 0$ and $M > 0$. A real number x is said to be the logarithm of M to the base a, denoted by $\log_a M$ if $a^x = M$

Thus $x = \log_a M$ and $a^x = M$ have the same meaning

Note that $a^{\log_a M} = M$.

Example $2^3 = 8$. So, $\log_2 8 = 3$

$3^4 = 81$. So, $\log_3 81 = 4$

$6^3 = 216$. So, $\log_6 216 = 3$

$10^3 = 1000$. So, $\log_{10} 1000 = 3$

$8^{1/3} = 2$. So, $\log_8 2 = 1/3$

$(81)^{1/4} = 3$, So, $\log_{81} 3 = \frac{1}{4}$

$(243)^{1/3} = 7$. So, $\log_{243} 7 = 1/3$

we can easily check that

$$\log_2 4 = \log_3 9 = \log_4 16 = \log_5 25 = \log_6 36 = 2$$

$$\log_2 8 = \log_3 27 = \log_4 64 = \log_5 125 = \log_6 216 = 3 \text{ etc.}$$

we also note that

$$2^{-2} = \frac{1}{4}. \text{ So, } \log_2 \frac{1}{4} = -2$$

$$2^{-4} = \frac{1}{16}. \text{ So, } \log_2 \frac{1}{16} = -4$$

$$5^{-3} = \frac{1}{125}. \text{ So, } \log_5 \frac{1}{125} = -3 \text{ etc.}$$

Notes 1. logarith is not defined when $a > 0$ and $M \leq 0$.

2. base a is also taken as positive, i.e., $a > 0$ and $a \neq 1$.

3. $\log_a 1 = 0$ for any real $a > 0$

4. $\log_a a = 1$ for $a > 0$

5.2 Laws of Logarithm

Result 1

$\log_a (M \times N) = \log_a M + \log_a N$ when $M > 0$, $N > 0$ and $a > 0$ and $a \neq 1$

Proof : Let $\log_a M = x$ and $\log_a N = y$