Then, by definition $a^x = x$ and $a^y = y$

$\Rightarrow$ $a^x = M$ and $a^y = N$

Now $M \times N = a^x \times a^y = a^{x+y}$

Since $M \times N > 0$, so, by definition of logarithm

$$\log_a M \times N = x + y = \log_a M + \log_a N$$

Result 2 $\log_a \frac{M}{N} = \log_a M - \log_a N$ where $M > 0$, $N > 0$ and $a > 0$ and $a \neq 1$.

Proof : Let $\log_a M = x$ and $\log_a N = y$

So, by definition,

$$a^x = M \quad \text{and} \quad a^y = N$$

Now, $$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}$$

Since $\frac{M}{N} > 0$, so, by ~~definition~~ definition of logarithm

$$\log_a \frac{M}{N} = x - y = \cancel{\log} \log_a M - \log_a N$$

Result 3 $\log_a M^p = p \log_a M$, $M > 0$, $a > 0$, $a \neq 1$ and p is real.

Proof: Let $\log_a M = x$

So, $a^x = M$

or, $(a^x)^p = M^p$

or, $a^{px} = M^p$. Since $M>0$, $M^p>0$

So, by definition of logarithm,

$$\log_a M^p = px = p\log_a M$$

<u>Result 4</u> $\log_a M = \log_b M \times \log_a b$, $M>0$, $a\neq 1$, $a>0$, $b>0$ and $b\neq 1$

Proof: Let $\log_a M = x$, $\log_b M = y$ and $\log_a b = z$

So, $a^x = M$, $b^y = M$ and $a^z = b$

So, $b^y = M$ gives $(a^z)^y = M$ or, $a^{yz} = M$

So, $M = a^x = a^{yz}$

So, $x = \log_a M = yz$

or, $x = yz$ or, $\log_a M = \log_b M \times \log_a b$

<u>Notes</u> 1. This law is applicable to change the base of a logarithm.

2. If $M = b$, then $\log_b b = 1$

3. If $M = a$, then $\log_b a \times \log_a b = \log_a a = 1$.

So, $\log_b a = \dfrac{1}{\log_a b}$

4. $\dfrac{\log_c a}{\log_c b} = \log_b a$

<u>Result 5</u> If $M < N$, then (i) $\log_a M < \log_a N$ for $a>1$ and $M>0$, $N>0$

(ii) $\log_a M > \log_a N$, for $0<a<1$ and $M>0$, $N>0$

Proof: (i) Let $\log_a M = x$ and $\log_a N = y$

So, $a^x = M$ and $a^y = N$

Now (i) $M < N$ is given; so $a^x < a^y$, $a > 1$

So, $x < y$ (by the property of indices)

or, $\log_a M < \log_a N$

(ii) Here $0 < a < 1$

and $M < N$ is given

So, $a^x < a^y$

or, $x > y$ (since $0 < a < 1$)

or, $\log_a M > \log_a N$

Example 1 Find the value of $\log_2 512$

Solution: $\log_2 512 = \log_2 2^9$

$= 9 \log_2 2 = 9 \times 1 = 9.$

Example 2 The logarithm of a positive number to the base $\sqrt{2}$ is a. What is its logarithm to the base $2\sqrt{2}$

Solution: Let the positive number be x.

So, $\log_{\sqrt{2}} x = a$. By definition of logarithm,

$$\left(\sqrt{2}\right)^a = x \quad \text{-----} \quad (1)$$

Now the base has to be changed ~~a~~ to $2\sqrt{2}$ with the same number x. We know that

$$2\sqrt{2} = \left(\sqrt{2}\right)^3 \quad \text{or,} \quad \left(2\sqrt{2}\right)^{1/3} = \sqrt{2}$$

$$\text{or,} \quad \left(2\sqrt{2}\right)^{a/3} = \left(\sqrt{2}\right)^a = x \quad \left[\text{from } (1)\right]$$

So, by definition of logarithm,

$$\log_{2\sqrt{2}} x = \frac{a}{3} .$$

Example 3 Prove that $\log(1+2+3) = \log 1 + \log 2 + \log 3$

(Here you can take any number $a > 0$ and $\neq 1$ as base)

Solution: $\log(1+2+3) = \log 6 = \log(1 \times 2 \times 3)$

$$= \log 1 + \log 2 + \log 3 .$$

Notes: 1. If in a context, all the base of a logarithm is equal, then sometimes we do not write the base.

2. In calculus, we use the base e, where the irrational number $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$ $(2 < e < 3)$

3. When we use the base 10, it is called natural logarithm.

Example 4 Find the value of

$$7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80}$$

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