Solution : $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$

$= 7\left[\log 16 - \log 15\right] + 5\left[\log 25 - \log 24\right] + 3\left[\log 81 - \log 80\right]$

$= 7\left[\log 2^4 - \log(3\times 5)\right] + 5\left[\log 5^2 - \log(2^3\times 3)\right] + 3\left[\log 3^4 - \log(2^4\times 5)\right]$

$= 7\left[4\log 2 - \log 3 - \log 5\right] + 5\left[2\log 5 - 3\log 2 - \log 3\right] + 3\left[4\log 3 - 4\log 2 - \log 5\right]$

$= \left[28\log 2 - 15\log 2 - 12\log 2\right] + \left[-7\log 3 - 5\log 3 + 12\log 3\right] + \left[-7\log 5 + 10\log 5 - 3\log 5\right]$

$= \log 2$

Example 5 If $x = \log_{bc} a$, $y = \log_{ca} b$ and $z = \log_{ab} c$

then show that $\frac{x}{1+x} + \frac{y}{1+y} + \frac{z}{1+z} = 1$

Solution: We are given that $x = \log_{bc} a$

or, $\frac{1}{x} = \frac{1}{\log_{bc} a} = \log_a bc$

or, $\frac{1}{x} + 1 = \log_a bc + 1 = \log_a bc + \log_a a$

or, $\frac{1+x}{x} = \log_a abc$

or, $\frac{x}{1+x} = \log_{abc} a$

Similarly, $\frac{y}{1+y} = \log_{abc} b$

and $\frac{z}{1+z} = \log_{abc} c$

So, $\frac{x}{1+x} + \frac{y}{1+y} + \frac{z}{1+z} = \log_{abc} a + \log_{abc} b + \log_{abc} c$

$= \log_{abc} abc$

$= 1$

Example 6 If $y = a^{\frac{1}{1-\log_a x}}$, $z = a^{\frac{1}{1-\log_a y}}$, then

show that, $x = a^{\frac{1}{1-\log_a z}}$.

Solution : Given that $y = a^{\frac{1}{1-\log_a x}}$. So, by

definition, $\log_a y = \frac{1}{1-\log_a x}$

or, $1 - \log_a x = \frac{1}{\log_a y}$

or, $\log_a x = 1 - \frac{1}{\log_a y}$

Similarly, by definition,

$\log_a z = \frac{1}{1-\log_a y}$

or, $1 - \log_a y = \frac{1}{\log_a z}$

or, $\log_a y = 1 - \frac{1}{\log_a z}$

Now, $\log_a x = 1 - \dfrac{1}{\log_a y}$

$$= 1 - \frac{1}{1 - \frac{1}{\log_a z}}$$

$$= 1 - \frac{\log_a z}{\log_a z - 1}$$

$$= \frac{-1}{\log_a z - 1} = \frac{1}{1 - \log_a z}$$

So, $x = a^{\frac{1}{1-\log_a z}}$

Example 7 Show that $\log_{10} 2$ lies between $\frac{1}{3}$ and $\frac{1}{4}$

Solution: $2^3 < 10 < 2^4$

or, $\log_a 2^3 < \log_a 10 < \log_a 2^4$ (Taking $a>1$)

or, $3 \log_a 2 < \log_a 10 < 4 \log_a 2$

So, $3 < \dfrac{\log_a 10}{\log_a 2} < 4$ $\quad (\because \log_a 2 > 0)$

or, $\dfrac{1}{3} > \dfrac{\log_a 2}{\log_a 10} > \dfrac{1}{4}$

or, $\dfrac{1}{4} < \log_{10} 2 < \dfrac{1}{3}$ $\quad \left(\because \dfrac{\log_a 2}{\log_a 10} = \log_{10} 2\right)$

Example 8 If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$,

Show that $x^x \cdot y^y \cdot z^z = 1$

Solution: Let $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = c$ (say)

Then $\log x = c(y-z)$ or, $x\log x = cx(y-z)$

$\log y = c(z-x)$ or, $y\log y = cy(z-x)$

and $\log z = c(x-y)$ or, $z\log z = cz(x-y)$

So, $x\log x + y\log y + z\log z = c(xy - xz + yz - xy + zx - yz)$

$= 0$

or, $\log x^x + \log y^y + \log z^z = 0$

or, $\log x^x \cdot y^y \cdot z^z = 0 = \log 1$

or, $x^x \cdot y^y \cdot z^z = 1$

Example 9 Solve for x : $\log_x(8x-3) - \log_x 4 = 2$

Solution: Given that $\log_x(8x-3) - \log_x 4 = 2$

or, $\log_x \frac{8x-3}{4} = 2$. Then from the definition of Logarithm,

$\frac{8x-3}{4} = x^2$ or, $4x^2 - 8x + 3 = 0$

or, $4x^2 - 6x - 2x + 3 = 0$

or, $2x(2x-3) - (2x-3) = 0$

or, $(2x-3)(2x-1) = 0$

or, $x = \frac{3}{2}$ or $\frac{1}{2}$