Example 10 If $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$, find x

Solution: Given that

$$\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$$

or, $\log_2 x + \frac{1}{\log_x 4} + \frac{1}{\log_x 16} = \frac{21}{4}$ $\left(\text{As } \log_b a = \frac{1}{\log_a b}\right)$

or, $\log_2 x + \frac{1}{2\log_x 2} + \frac{1}{4\log_x 2} = \frac{21}{4}$

or, $\log_2 x + \frac{1}{2}\log_2 x + \frac{1}{4}\log_2 x = \frac{21}{4}$

or, $\left(1 + \frac{1}{2} + \frac{1}{4}\right)\log_2 x = \frac{21}{4}$

or, $\frac{7}{4}\log_2 x = \frac{21}{4}$

or, $\log_2 x = 3$

Hence $x = 2^3 = 8$

Example 11 The first and last terms of a G.P. are $a(>0)$ and $k(>0)$ respectively. If the number of terms be n, prove that $n = 1 + \frac{\log k - \log a}{\log r}$, where r is the common ratio.

Solution: Let the G.P. series be $a, ar, ar^2, \ldots, ar^{n-1}$. The last term or the nth term $= k = ar^{n-1}$

Now taking logarithm to both sides, we have

$$\log k = \log(ar^{n-1}) = \log a + \log r^{n-1}$$
$$= \log a + (n-1)\log r$$

or, $\log k - \log a = (n-1)\log r$

or, $n-1 = \dfrac{\log k - \log a}{\log r}$

or, $n = 1 + \dfrac{\log k - \log a}{\log r}$

Example 12 If $\log_p x = a$ and $\log_q x = b$, then prove that $\log_{p/q} x = \dfrac{ab}{b-a}$.

Solution : It is given that

$$\log_p x = a$$

or, $\log_x p = \dfrac{1}{a}$ $\left[\text{as } \log_b a = \dfrac{1}{\log_a b}\right]$

and $\log_q x = b$

or, $\log_x q = \dfrac{1}{b}$

Now $\log_{p/q} x = \dfrac{1}{\log_x p/q} = \dfrac{1}{\log_x p - \log_x q}$

$$= \frac{1}{\frac{1}{a} - \frac{1}{b}} = \frac{ab}{b-a}$$

So, $\log_{p/q} x = \dfrac{ab}{b-a}$

Example 13 Find the value of $\dfrac{\log\sqrt{27} + \log 8 + \log\sqrt{1000}}{\log 120}$

(Note: In algebra, if we write $\log x$, then it is $\log_{10} x$ and in calculus if we write $\log x$, then it is $\log_e x$ where e is the irrational number given by

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots)$$

Solution: The given expression is

$$\frac{\log\sqrt{27} + \log 8 + \log\sqrt{1000}}{\log 120}$$

$$= \frac{\log 3^{3/2} + \log 4^{3/2} + \log 10^{3/2}}{\log 120}$$

$$= \frac{\frac{3}{2}\left(\log 3 + \log 4 + \log 10\right)}{\log 120}$$

$$= \frac{\frac{3}{2}\log 3\times 4\times 10}{\log 120}$$

$$= \frac{\frac{3}{2}\log 120}{\log 120} = \frac{3}{2}$$

Example 14 ~~Prove that~~

Prove that $\log_2\log_2\log_2 16 = 1$

Solution: $\log_2\log_2\log_2 16$

$$= \log_2\log_2\log_2 2^4$$

$$= \log_2\log_2 4\log_2 2$$

$$= \log_2\log_2 4 \qquad [\because \log_2 2 = 1]$$

$= \log_2 2 \log_2 2 = \log_2 2 = 1$

Example 15 If $\log_a bc = x$, $\log_b ca = y$ and $\log_c ab = z$, then show that $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1$

Solution: $L.H.S = \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}$

$$= \frac{1}{1+\log_a bc} + \frac{1}{1+\log_b ca} + \frac{1}{1+\log_c ab}$$

$$= \frac{1}{\log_a a + \log_a bc} + \frac{1}{\log_b b + \log_b ca} + \frac{1}{\log_c c + \log_c ab}$$

$$= \frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc}$$

$$= \log_{abc} a + \log_{abc} b + \log_{abc} c$$

$$= \log_{abc} abc = 1 = R.H.S$$

Example 16 Prove that $x^{\log y - \log z} \cdot y^{\log z - \log x} \cdot z^{\log x - \log y} = 1$

Solution Let $A = x^{\log y - \log z} \cdot y^{\log z - \log x} \cdot z^{\log x - \log y}$

Now, $\log A = \log x^{\log y - \log z} + \log y^{\log z - \log x} + \log z^{\log x - \log y}$

$= (\log y - \log z)\log x + (\log z - \log x)\log y + (\log x - \log y)\log z$

$= \log x \log y - \log x \log z + \log y \log z - \log x \log y + \log x \log z - \log y \log z$

$= 0$

So, $\log A = 0 = \log 1$. So $A = 1$

So, $x^{\log y - \log z} \cdot y^{\log z - \log x} \cdot z^{\log x - \log y} = 1$