6. Identity Laws

(i) $A \cup \phi = A$ (ii) $A \cap \phi = A$

(iii) $A \cup S = S$ (iv) $A \cap S = A$

7. Complement Laws

(i) $A \cup A' = S$ (ii) $A \cap A' = \phi$

(iii) $(A')' = A$ (iv) $S' = \phi$, $\phi' = S$

1.7 Proofs of the Laws of Algebra of sets.

1. In order to prove that $A \subseteq B$ we show that $x \in A \Rightarrow x \in B$ ('$\Rightarrow$' means implies)

So, $A \subseteq B$ if $x \in A \Rightarrow x \in B$

2. In order to show that $A = B$ we have to show that $A \subseteq B$ and $B \subseteq A$

Examp 6 (commutative laws)

For any two sets A and B, prove that

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

Proof: (i) $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in B \cup A$

So, $A \cup B \subseteq B \cup A$

$y \in B \cup A \Rightarrow y \in B$ or $y \in A$

$\Rightarrow y \in A$ or $y \in B$

$\Rightarrow y \in A \cup B$

So, $B \cup A \subseteq A \cup B$

So, $A \cup B = B \cup A$

(ii) $x \in A \cap B \Rightarrow x \in A$ and $x \in B$
$\Rightarrow x \in B$ and $x \in A$
$\Rightarrow x \in B \cap A$

So, $A \cap B \subseteq B \cap A$

$y \in B \cap A \Rightarrow y \in B$ and $y \in A$
$\Rightarrow y \in A$ and $y \in B$
$\Rightarrow y \in A \cap B$

So, $B \cap A \subseteq A \cap B$

So, $A \cap B = B \cap A$

Example 7 (Associative Laws)

For any three sets A, B and C

(i) $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

Proof: (i) $x \in A \cup (B \cup C) \Rightarrow x \in A$ or $x \in (B \cup C)$
$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$
$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$
$\Rightarrow x \in (A \cup B)$ or $x \in C$
$\Rightarrow x \in (A \cup B) \cup C$

So, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

$y \in (A \cup B) \cup C \Rightarrow y \in (A \cup B)$ or $y \in C$
$\Rightarrow (y \in A$ or $y \in B)$ or $y \in C$
$\Rightarrow y \in A$ or $(y \in B$ or $y \in C)$
$\Rightarrow y \in A$ or $y \in (B \cup C)$
$\Rightarrow y \in A \cup (B \cup C)$

So, $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

So, $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) $x \in A \cap (B \cap C) \Rightarrow x \in A$ and $x \in (B \cap C)$

$\Rightarrow x \in A$ and $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow x \in (A \cap B)$ and $x \in C$

$\Rightarrow x \in (A \cap B) \cap C$

So, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$

$y \in (A \cap B) \cap C \Rightarrow y \in A \cap B$ and $y \in C$

$\Rightarrow (y \in A$ and $y \in B)$ and $y \in C$

$\Rightarrow y \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow y \in A$ and $y \in (B \cap C)$

$\Rightarrow y \in A \cap (B \cap C)$

So, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

So, $A \cap (B \cap C) = (A \cap B) \cap C$

Example 8 (Distributive Laws)

For any three sets A, B and C, prove that

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

So, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

$y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow (y \in A$ or $y \in B)$ and $(y \in A$ or $y \in C)$

$\Rightarrow y \in A$ or $(y \in B$ and $y \in C)$

$\Rightarrow y \in A$ or $y \in (B \cap C)$

$\Rightarrow y \in A \cup (B \cap C)$

So, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

So, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in B \cup C$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

So, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

$y \in (A \cap B) \cup (A \cap C) \Rightarrow y \in (A \cap B)$ or $y \in (A \cap C)$

$\Rightarrow (y \in A$ and $y \in B)$ or $(y \in A$ and $y \in C)$

$\Rightarrow y \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow y \in A$ and $y\,(B \cup C)$

$\Rightarrow y \in A \cap (B \cup C)$

So, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

So, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example 9 (De Morgan's laws)

For any two sets A and B, prove that

(i) $(A \cup B)^c = A^c \cap B^c$

(ii) $(A \cap B)^c = A^c \cup B^c$, A^c = complement of A

Proof: $x \in (A \cup B)^c \Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$