Example 16 Prove that

(i) $A \cap (B - C) = (A \cap B) - C$

(ii) $A \cap (B - C) = (A \cap B) - (A \cap C)$

Solution: (i) $x \in A \cap (B - C) \Rightarrow x \in A$ and $x \in (B - C)$

$\Rightarrow x \in A$ and $(x \in B$ and $x \notin C)$

$\Rightarrow (x \in A$ and $x \in B)$ and $x \notin C$

$\Rightarrow x \in A \cap B$ and $x \notin C$

$\Rightarrow x \in (A \cap B) - C$

So, $A \cap (B - C) \subseteq (A \cap B) - C \quad \ldots (1)$

Now $y \in (A \cap B) - C \Rightarrow y \in (A \cap B)$ and $y \notin C$

$\Rightarrow (y \in A$ and $y \in B)$ and $y \notin C$

$\Rightarrow y \in A$ and $(y \in B$ and $y \notin C)$

$\Rightarrow y \in A$ and $y \in B - C$

$\Rightarrow y \in A \cap (B - C)$

So, $(A \cap B) - C \subseteq A \cap (B - C) \quad \ldots (2)$

Hence from (1) and (2), we have

$A \cap (B - C) = (A \cap B) - C$

(ii) R.H.S $= (A \cap B) - (A \cap C)$

$= (A \cap B) \cap (A \cap C)'$ since $X - Y = X \cap Y'$

$= (A \cap B) \cap (A' \cup C')$ by De Morgan's Law

$= \{(A \cap B) \cap A'\} \cup \{(A \cap B) \cap C'\}$ by Distributive Law

$= \{A' \cap (A \cap B)\} \cup \{(A \cap B) \cap C'\}$ by Commutative Law

$= \{(A' \cap A) \cap B\} \cup \{A \cap (B \cap C')\}$ by Associative Law

$= \{(A \cap A') \cap B\} \cup \{A \cap (B \cap C')\}$ by Commutative Law

$= (\phi \cap B) \cup \{A \cap (B \cap C')\}$ by Complement Law

$= \phi \cup \{A \cap (B \cup C')\}$ by Identity Law

$= A \cap (B \cup C')$ by Identity Law

$= A \cap (B - C)$ Since $B - C = B \cap C'$

$=$ ~~R.H.S~~ L.H.S

So, $A \cap (B-C) = (A \cap B) - (A \cap C)$.

Example 17 Using set algebra, show that

(i) $A - (B \cap C) = (A-B) \cup (A-C)$

(ii) $(A \cup B) - C = (A-C) \cup (B-C)$

(iii) $(A-B) \cup (B-A) = (A \cup B) - (A \cap B)$

Solution : (i) $A - (B \cap C) = A \cap (B \cap C)'$ Since $A - D = A \cap D'$

$= A \cap (B' \cup C')$ by De Morgan's Law

$= (A \cap B') \cup (A \cap C')$ by Distributive Law

$= (A-B) \cup (A-C)$ Since $A - B = A \cap B'$

(ii) $(A \cup B) - C = (A \cup B) \cap C'$ Since $A - B = A \cap B'$

$= (A \cap C') \cup (B \cap C')$ by Distributive Law

$= (A-C) \cup (B-C)$ Since $A - B = A \cap B'$

(iii) $(A-B)\cup(B-A) = (A\cap B')\cup(B\cap A')$ Since $A-B = A\cap B'$

$= \{A\cup(B\cap A')\}\cap\{B'\cup(B\cap A')\}$ by Distributive Law

$= \{(A\cup B)\cap(A\cup A')\}\cap\{(B'\cup B)\cap(B'\cup A')\}$ by Distributive Law

$= \{(A\cup B)\cap S\}\cap\{S\cap(B'\cup A')\}$ by Complement Law

by Identity Law

$= (A\cup B)\cap(B'\cup A')$ By Commutative Law

$= (A\cup B)\cap(A'\cup B')$ by De Morgan's Law

$= (A\cup B)\cap(A\cap B)'$

$= (A\cup B) - (A\cap B)$ Since $C\cap D' = C-D$

Cartesian Product of Sets (Important results)

1. If $A\subseteq C$ then $A\times B\subseteq C\times B$
   If $A\subseteq C$ and $B\subseteq D$ then $A\times B\subseteq C\times D$

2. (i) $A\times(B\cup C) = (A\times B)\cup(A\times C)$
   (ii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$

3. $A\times(B-C) = (A\times B)-(A\times C)$

4. $(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D)$

Example 18 If $A\subseteq C$ and $B\subseteq D$ Show that $A\times B\subseteq C\times D$

$(a,b)\in A\times B \Rightarrow a\in A$ and $b\in B$

$\Rightarrow a\in C$ and $b\in D$ $(\because A\subseteq C$ and $B\subseteq D)$

$\Rightarrow (a,b)\in C\times D$

So, $A\times B\subseteq C\times D$

Example 19 Prove that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

Solution: (i) $(x,y) \in A \times (B \cup C)$

$\Rightarrow x \in A$ and $y \in (B \cup C)$

$\Rightarrow x \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$.

$\Rightarrow (x,y) \in A \times B$ or $(x,y) \in A \times C$

$\Rightarrow (x,y) \in (A \times B) \cup (A \times C)$

$\Rightarrow (x,y) \in (A \times B) \cup (A \times C) \quad \cdots (1)$.

So, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

$(a,b) \in (A \times B) \cup (A \times C)$

$\cancel{\Rightarrow a \in A \times B \text{ or}}$

$\Rightarrow (a,b) \in (A \times B)$ or $(a,b) \in A \times C$

$\Rightarrow (a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$

$\Rightarrow a \in A$ and $(b \in B$ or $b \in C)$

$\Rightarrow a \in A$ and $b \in B \cup C$

$\Rightarrow (a,b) \in A \times (B \cup C)$

So, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C) \quad \cdots (2)$

Hence from (1) and (2), we have

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$