12. How many 5-digit odd numbers can be made with 3, 2, 7, 4 and 0 without repeating the digits?

13. How many arrangements can ~~be made~~ be made out of the letters of the word 'COMMITTEE', taken all at a time such that 4 vowels do not come together?

$\boxed{\text{COMBINATION}}$

2.1 Introduction

The different groups or selections that can be formed out of a given set of objects by taking some or all of them at a time without considering their arrangements is known as combination. Combination may be looked upon as a subset of a set of objects. The basic difference between combination and permutation is that when we consider different permutations from a set of objects, we take into account their order of appearances in different arrangements, but in case of combinations we ignore the order of appearance of the objects.

For example, let us consider three letters a, b, and c. Taking three at a time, we have only one group or selection, which is abc, whereas it was observed earlier in the discussion of permutation that the number of distinct permutations is six. Again if we take two letters at a time out of three then the combinations are : ab, ac, and bc i.e., there are three combinations but we have observed earlier that there six distinct permutation.

If we want to make a choice of three letters from four distinct letters a, b, c, and d then we are having four choices abc, abd, acd and bcd whereas the total number of permutations is $4! = 24$.

Then, it is clearly understood that the combination is nothing but formation of group or selection, whereas in permutation in addition to selection, the order of selection is also of great importance.

2.2 Results of Combination

Result 2.2.1 The number of combination of n distinct objects taken $r (\leq n)$ at a time is given by (denoted by ${}^{n}C_{r}$)

$$ {}^{n}C_{r} = \frac{n!}{r!(n-r)!} $$

Proof : Suppose x be the number of combinations of n objects selected r at a time. By the given notation, $x = {}^{n}C_{r}$. Further each of the combinations consists of r objects and these r objects can be arranged in $r!$ ways among themselves. Therefore, the total permutation will be $x \cdot r!$. But from the notation of permutation $x \cdot r! = {}^nP_r$

or, $x = \dfrac{{}^nP_r}{r!} = \dfrac{n!}{r!\,(n-r)!}$

Hence, ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$

<u>Some important properties of combination</u>

1. $${}^nC_r = \frac{n!}{r!\,(n-r)!}$$
$$= \frac{n(n-1)\cdots(n-r+1)\cdot(n-r)!}{r!\,(n-r)!}$$
$$= \frac{n(n-1)\cdots(n-r+1)}{r!}$$

2. ${}^nP_r = r!\,{}^nC_r$, that is, ${}^nP_r \geq {}^nC_r$

3. ${}^nC_n = \dfrac{n!}{n!\,(n-n)!} = \dfrac{1}{0!} = 1 \quad (\because 0! = 1)$

${}^nC_0 = \dfrac{n!}{0!\,(n-0)!} = 1$. So, ${}^nC_n = {}^nC_0$

4. ${}^nC_1 = \dfrac{n!}{1!\,(n-1)!} = n$. Again, ${}^nC_{n-1} = \dfrac{n!}{(n-1)!\,(n-(n-1))!}$
$$= \frac{n!}{(n-1)!\,1!} = n$$

So, ${}^nC_{n-1} = {}^nC_1$

5. In general ${}^nC_r = {}^nC_{n-r}$, $0 \leq r \leq n$

Proof: ${}^nC_r = \dfrac{n!}{r!\,(n-r)!} = \dfrac{n!}{(n-r)!\,(n-(n-r))!} = {}^nC_{n-r}$

6. $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}, \quad 1 \le r \le n$

Proof: $^nC_r + {^nC_{r-1}}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{(n+1)!}{r!(n+1-r)!} = {^{n+1}C_r}$$

7. $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

Proof

$$\frac{^nC_r}{^nC_{r-1}} = \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r-1)!(n-r+1)!}} = \frac{(r-1)!(n-r+1)!}{r!(n-r)!} = \frac{n-r+1}{r}$$

8. $r \cdot {^nC_r} = n \cdot {^{n-1}C_{r-1}}$

Proof:

$$\frac{^nC_r}{^{n-1}C_{r-1}} = \frac{\frac{n!}{r!(n-r)!}}{\frac{(n-1)!}{(r-1)!\{(n-1)-(r-1)\}!}} = \frac{n!(r-1)!(n-r)!}{(n-1)!\,r!(n-r)!} = \frac{n}{r}$$

So, $r \cdot {^nC_r} = n \cdot {^{n-1}C_{r-1}}$.