Some restricted combinations

Result 2.3

The number of combinations of n distinct objects taken r at a time in which

(a) q particular objects always occur is given by $^{n-q}C_{r-q}$, $q < r \leq n$

(b) q particular objects never occur is given by $^{n-q}C_r$, $r \leq n$

Result 2.4

The total number of combinations of n different objects taken $1, 2, 3, \ldots, n$ at a time is given by

$$^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n - 1$$

Result 2.5

The total number of combinations of $(p+q+r+\cdots)$ objects in which p objects are alike of first kind, q objects alike of second kind, r objects alike of third kind and so on is given by $[(p+1)(q+1)(r+1)\cdots] - 1$

Note If $p = q = r = \cdots = 1$, then we get result 2.4

Result 2.6

The number of combinations of $(p+q)$ objects from $(m+n)$ distinct objects in such a way that p objects are selected from m distinct objects and q objects are selected from another n distinct object is given by

$$^mC_p \times {}^nC_q \quad \text{where } m \geq p,\ n \geq q$$

Result 2.7

The number of ways in which $(m+n)$ distinct objects can be divided into two groups containing m and n objects, respectively, is given by

$$^{m+n}C_m \times {}^{n}C_n = \frac{(m+n)!}{m!\,n!}, \quad m \neq n$$

Note If $m=n$, then out of $^{2n}C_n$ groups, ~~as identical~~ $\frac{1}{2} \times {}^{2n}C_n$ are identical with the rest.

Therefore, the number of groups $= \frac{1}{2} \times \frac{2n!}{(n!)^2}$

Further, if 2n objects be divided equally between two persons, then the number of different ways would be $\frac{2n!}{(n!)^2}$

Result 2.8

The number of ways in which $(m+n+p)$ distinct objects can be divided into three groups containing m, n and p items, respectively, is given by

$$^{m+n+p}C_m \times {}^{n+p}C_n \times {}^{p}C_p = \frac{(m+n+p)!}{m!\,n!\,p!}, \quad m \neq n \neq p.$$

Note If $m=n=p$, then each of the three groups contains the same number of objects and the three groups can be interchanged among themselves by 3! ways. Therefore, the number of groups

$$= \frac{(3p)!}{3!\,(p!)^3}$$

Further 3p objects are equally divided among three persons, then the number of different groups would be $= \frac{(3p)!}{(p!)^3}$

Example 1 Find the value of (a) ${}^{12}C_5$ (b) ${}^{15}C_{13}$ (c) ${}^{8}C_5 + {}^{8}C_4$.

Solution: (a) $${}^{12}C_5 = \frac{12!}{5!\,(12-5)!} = \frac{12!}{5!\,7!} = \frac{12\times11\times10\times9\times8}{5\times4\times3\times2\times1} = 792$$

(b) $${}^{15}C_{13} = {}^{15}C_2 = \frac{15!}{2!\,(15-2)!} = \frac{15!}{2!\,13!} = \frac{15\times14}{2\times1} = 105$$

(c) $${}^{8}C_5 + {}^{8}C_4 = {}^{9}C_5 \quad \left(\because {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r\right)$$

$$= \frac{9!}{5!\,(9-5)!} = \frac{9!}{5!\,4!} = \frac{9\times8\times7\times6}{4\times3\times2\times1} = 126$$

Example 2 Find n if ${}^{n}C_2 = 21$.

Solution: Given that ${}^{n}C_2 = 21$

or, $\frac{n(n-1)}{2} = 21$

or, $n^2 - n - 42 = 0$

or, $(n-7)(n+6) = 0$

Since n can not be a negative integer, $n = 7$.

Example 3 Find ${}^{2n}C_3$ if ${}^{n}C_8 = {}^{n}C_6$

Solution It is given that: ${}^{n}C_8 = {}^{n}C_6$

So, $${}^{n}C_8 = {}^{n}C_{n-6} \quad \left(\because {}^{n}C_r = {}^{n}C_{n-r}\right)$$

or, $n - 6 = 8$

or, $n = 14$

Hence $${}^{2n}C_3 = {}^{28}C_3 = \frac{28!}{3!\,(28-3)!} = \frac{28!}{3!\,25!}$$

$$= \frac{28\times27\times26}{3\times2\times1} = 3276$$

Example 4 Find n and r if ${}^{n}C_{r-1} : {}^{n}C_r : {}^{n}C_{r+1} = 3:4:5$

Solution: We are given that

$$\frac{{}^{n}c_{r-1}}{3} = \frac{{}^{n}c_{r}}{4} = \frac{{}^{n}c_{r+1}}{5}$$

So, $\dfrac{{}^{n}c_{r-1}}{3} = \dfrac{{}^{n}c_{r}}{4}$ gives

$$\text{or,} \quad \frac{\dfrac{n!}{(r-1)!\,(n-r+1)!}}{3} = \frac{\dfrac{n!}{r!\,(n-r)!}}{4}$$

$$\text{or,} \quad \frac{1}{3(n-r+1)} = \frac{1}{4r}$$

$$\text{or,} \quad 3n-3r+3 = 4r$$

$$\text{or,} \quad 3n-7r+3 = 0 \quad \ldots \quad (1)$$

Again, $\dfrac{{}^{n}c_{r}}{4} = \dfrac{{}^{n}c_{r+1}}{5}$ gives

$$\frac{\dfrac{n!}{r!\,(n-r)!}}{4} = \frac{\dfrac{n!}{(r+1)!\,(n-r-1)!}}{5}$$

$$\text{or,} \quad \frac{1}{4(n-r)} = \frac{1}{5(r+1)}$$

$$\text{or,} \quad 5r+5 = 4n-4r$$

$$\text{or,} \quad 4n-9r-5 = 0 \quad \ldots \quad (2)$$

So, from (1) and (2), we have

$$\frac{n}{35+27} = \frac{r}{12+15} = \frac{1}{-27+28}$$

$$\text{or,} \quad n = 62, \quad r = 27$$

Example 5 From 7 boys and 8 girls, 5 persons are to be selected. In how many ways, the selection can be done if exactly two boys must be there?