~~(i) $X' = I^{-1}X = I^{-1}A$, (ii) $X = SX'$ and (iii) $X = SX' + A$ respectively~~

(i) $X' = I^{-1}X - I^{-1}A$, (ii) $X' = S^{-1}X$ and (iii) $X' = S^{-1}X - S^{-1}A$.

I^{-1} and S^{-1} are inverses of the matrices I and S

Then $I^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Note 1. I and S are ~~a~~ orthogonal matrices and $|I| = 1 = |S|$. For this property of I and S each transformation is called an orthogonal transformation.

Note 2 The transformation by the formulae

$$\left.\begin{aligned} x &= \lambda x' - \mu y' + h \\ y &= \mu x' + \lambda y' + k \end{aligned}\right\} \text{ where } \lambda^2 + \mu^2 = 1$$

is an orthogonal transformation.

If $\lambda = 1, \mu = 0$, then it is translation

If $h = 0 = k$ then it is translation.

1.5 Invariants. Some expressions remain unchanged under an orthogonal transformation. These are known as invariants of orthogonal transformation.

(i) The degree of an equation is an invariant under ~~a~~ orthogonal transformation

~~Proof Let (x_1, y_1) and (x_2, y_2) be the coordinates of two~~

Proof: Let a polynomial equation be $f(x,y)=0$

By an orthogonal transformation of the co-ordinates axes the equation transforms to $f(\lambda x'-\mu y'+h, \mu x'+\lambda y'+k)=0$ where $\lambda^2+\mu^2=1$. If $ax^p y^q$ is the highest degree term in $f(x,y)=0$, then in the transformed equation, it becomes $a(\lambda x'-\mu y+h)^p(\mu x'+\lambda y'+k)^q$. The highest degree term containing x' or $x'y'$ or y' is $p+q$. It is equal to the degree of $ax^p y^q$. Thus the degree of an equation remains invariant under orthogonal transformation

(ii) The distance between two points is an invariant under an orthogonal transformation.

Proof: Let (x_1, y_1) and (x_2, y_2) be the coordinates of two points in the old system, In the new system the coordinates of these points are (x_1', y_1') and (x_2', y_2').

The formulae for transformations are

$$\left.\begin{aligned} x &= \lambda x' - \mu y' + h \\ y &= \mu x' + \lambda y' + k \end{aligned}\right\} \text{ where } \lambda^2+\mu^2=1$$

If d is the distance between the points then

$$d^2 = (x_2-x_1)^2+(y_2-y_1)^2$$

By transformation, $(x_2-x_1)^2+(y_2-y_1)^2$

$= \{(\lambda x_2' - \mu y_2' + h) - (\lambda x_1' - \mu y_1' + h)\}^2 + \{(\mu x_2' + \lambda y_2' + k) - (\mu x_1' + \lambda y_1' + k)\}^2$

$= \{\lambda(x_2' - x_1') - \mu(y_2' - y_1')\}^2 + \{\mu(x_2' - x_1') + \lambda(y_2' - y_1')\}^2$

$= (\lambda^2 + \mu^2)\{(x_2' - x_1')^2 + (y_2' - y_1')^2\} = (x_2' - x_1')^2 + (y_2' - y_1')^2$

as $\lambda^2 + \mu^2 = 1$

$\therefore d^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2$. Hence the result follows.

(iii) The coefficients of x^2, xy, y^2 and $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

obtained from $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ are invariant under translation

Let the origin be shifted to (α, β). The expression transforms to

$a(x' + \alpha)^2 + 2h(x' + \alpha)(y' + \beta) + b(y' + \beta)^2 + 2g(x' + \alpha) + 2f(y' + \beta) + c$

$= ax'^2 + 2hx'y' + by'^2 + 2(a\alpha + h\beta + g)x' + 2(h\alpha + b\beta + f)y'$
$+ a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$

$= a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c'$ (say)

Here $a' = a$, $b' = b$, $h' = h$, $g' = a\alpha + h\beta + g$,

$f' = h\alpha + b\beta + f$ $\quad c' = a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c$

We see that the coefficients of x^2, y^2 and xy i.e., a, b and h remain invariant due to translation. Let us consider the invariance of $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

After translation Δ ~~chan~~ changes to Δ' (say)

now $\Delta' = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}$

$$= \begin{vmatrix} a & h & a\alpha + h\beta + g \\ h & b & h\alpha + b\beta + f \\ a\alpha + h\beta + g & h\alpha + b\beta + f & a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c \end{vmatrix}$$

$$= \begin{vmatrix} a & h & g \\ h & b & f \\ a\alpha + h\beta + g & h\alpha + b\beta + f & g\alpha + f\beta + c \end{vmatrix}$$

(on subtracting α times the elements of the first column and β times the elements of 2nd column from the elements of the third column)

$$= \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$ (On subtracting the α times of first row and β times the second row from the third row)

$= \Delta$

So, Δ is also invariant

(iv) $a+b$, $ab-h^2$, f^2+g^2 and Δ obtained from