$ax^2+2hxy+by^2+2gx+2fy+c$ remain invariant under transformation of rotation.

Proof: Let the axes be rotated through an angle θ then

$x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$, the expression

$ax^2+2hxy+by^2+2gx+2fy+c$ transforms to

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)^2$$
$$+2g(x'\cos\theta - y'\sin\theta) + 2f(x'\sin\theta + y'\cos\theta) + c$$

$$= a'x'^2 + b'y'^2 + 2h'x'y' + 2g'x' + 2f'y' + c' \text{ (say)}$$

Then $a' = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$

$b' = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$

$h' = (b-a)\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta)$

$g' = g\cos\theta + f\sin\theta$, $f' = -g\sin\theta + f\cos\theta$, $c' = c$

If we put $a_1 = a\cos\theta + h\sin\theta$, $b_1 = h\cos\theta + b\sin\theta$

$a_2 = -a\sin\theta + h\cos\theta$, $b_2 = -h\sin\theta + b\cos\theta$

then $a' = a_1\cos\theta + b_1\sin\theta$, $b' = -a_2\sin\theta + b_2\cos\theta$

$h' = a_2\cos\theta + b_2\sin\theta = -a_1\sin\theta + b_1\cos\theta$

[a_2, b_2 are derivatives of a_1 and b_1 with respect to θ]

(1) $a' + b' = a(\cos^2\theta + \sin^2\theta) + 2h\sin\theta\cos\theta - 2h\sin\theta\cos\theta + b(\sin^2\theta + \cos^2\theta)$

$= a + b$

(2) $$a'b' - h'^2 = \begin{vmatrix} a' & h' \\ h' & b' \end{vmatrix} = \begin{vmatrix} a_1\cos\theta + b_1\sin\theta & -a_1\sin\theta + b_1\cos\theta \\ a_2\cos\theta + b_2\sin\theta & -a_2\sin\theta + b_2\cos\theta \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a\cos\theta + h\sin\theta & h\cos\theta + b\sin\theta \\ -a\sin\theta + h\cos\theta & -h\sin\theta + b\cos\theta \end{vmatrix}$$

$$= \begin{vmatrix} a & h \\ h & b \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$= \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$$

(3) $g'^2 + f'^2 = (g\cos\theta + f\sin\theta)^2 + (-g\sin\theta + f\cos\theta)^2$

$$= g^2(\cos^2\theta + \sin^2\theta) + f^2(\sin^2\theta + \cos^2\theta)$$

$$= g^2 + f^2$$

(4) $\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = g' \begin{vmatrix} h' & g' \\ b' & f' \end{vmatrix} - f' \begin{vmatrix} a' & g' \\ \cancel{g}h' & f' \end{vmatrix} + c' \begin{vmatrix} a' & h' \\ h' & b' \end{vmatrix}$

$$= g' \begin{vmatrix} a_2\cos\theta + b_2\sin\theta & g\cos\theta + f\sin\theta \\ -a_2\sin\theta + b_2\cos\theta & -g\sin\theta + f\cos\theta \end{vmatrix}$$

$$- f' \begin{vmatrix} a_1\cos\theta + b_1\sin\theta & g\cos\theta + f\sin\theta \\ -a_1\sin\theta + b_1\cos\theta & -g\sin\theta + f\cos\theta \end{vmatrix} + c(ab - h^2) \quad \text{(By (2))}$$

$$= g' \begin{vmatrix} a_2 & b_2 \\ g & f \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} - f' \begin{vmatrix} a_1 & b_1 \\ g & f \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} + c(ab - h^2)$$

$$= g'\begin{vmatrix} a_2 & b_2 \\ g & f \end{vmatrix} - f'\begin{vmatrix} a_1 & b_1 \\ g & f \end{vmatrix} + c(ab-h^2)$$

$$= g'(a_2 f - b_2 g) - f'(a_1 f - b_1 g) + c(ab-h^2)$$

$$= g(b_1 f' - b_2 g') - f(a_1 f' - g' a_2) + c(ab-h^2)$$

$$= g\begin{vmatrix} b_1 & b_2 \\ g' & f' \end{vmatrix} - f\begin{vmatrix} a_1 & a_2 \\ g' & f' \end{vmatrix} + c(ab-h^2)$$

$$= g\begin{vmatrix} h\cos\theta + b\sin\theta & -h\sin\theta + b\cos\theta \\ g\cos\theta + f\sin\theta & -g\sin\theta + f\cos\theta \end{vmatrix}$$

$$- f\begin{vmatrix} a\cos\theta + h\sin\theta & -a\sin\theta + h\cos\theta \\ g\cos\theta + f\sin\theta & -g\sin\theta + f\cos\theta \end{vmatrix} + c(ab-h^2)$$

$$= g\begin{vmatrix} h & b \\ g & f \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} - f\begin{vmatrix} a & h \\ g & f \end{vmatrix} \times \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} + c(ab-h^2)$$

$$= g\begin{vmatrix} h & b \\ g & f \end{vmatrix} - f\begin{vmatrix} a & h \\ g & f \end{vmatrix} + c\begin{vmatrix} a & h \\ h & b \end{vmatrix}$$

$$= \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \Delta$$

So, $a+b$, $ab-h^2$, g^2+f^2 and Δ are invariant

Worked out examples

1. The equation $3x^2+2xy+3y^2-18x-22y+50=0$ is reduced to $4x^2+2y^2=1$ when referred to rectangular axes through the point $(2,3)$. Find the inclination of the latter axes to the former.

Solution: Due to translation, the equation transforms to

$$3(x'+2)^2+2(x'+2)(y'+3)+3(y'+3)^2-18(x'+3)-22(y'+3)+50=0$$

or, $3x'^2+2x'y'+3y'^2=1$

Let the axes be rotated through an angle θ. The equation changes to

$$3(x''\cos\theta-y''\sin\theta)^2+2(x''\cos\theta-y''\sin\theta)(x''\sin\theta+y''\cos\theta)+3(x''\sin\theta+y''\cos\theta)^2=1$$

or $(3+\sin 2\theta)x''^2+2\cos 2\theta\, x''y''+(3-\sin 2\theta)y''^2=1$

To remove the term $x''y''$ we take $\cos 2\theta=0$

or $2\theta=\pi/2$ or, $\theta=\pi/4$. For this value of θ

the equation reduces to $4x''^2+2y''^2=1$

So, the required inclination is $\pi/4$.

2. The coordinates of new origin are $(2,1)$ and the axes are rotated through angle 60°. If the coordinates of a point in the new system are $\left(\frac{3-4\sqrt{3}}{2}, \frac{4+3\sqrt{3}}{2}\right)$, find