called ~~too~~ canonical or normal form of the conic.

To find the canonical form from the equation (1), the following transformations are made successively :

(i) The term in xy is removed by suitable rotation of axes.

(ii) One or both (when possible) the terms in x and y are removed by translation.

(iii) The constant is removed, if possible.

This is called reduction to canonical form or normal form.

[Note: When $\Delta = 0$ and $D \neq 0$, the equation (1) represents a pair of straight lines (imaginary or intersecting) or a point ellipse. When $\Delta = 0$ and $D = 0$, the equation (1) represents a pair of parallel or coincident straight lines.]

Case 1 Equation (1) is a central conic (Ellipse or hyperbola).

Here $\Delta \neq 0$ and $D \neq 0$.

By transferring the origin to the centre (α, β) where, $a\alpha + h\beta + g = 0$ and $h\alpha + b\beta + f = 0$, the equation (1) reduces to $ax^2 + 2hxy + by^2 + d = 0$ ... (4)
(terms in x and y are removed)

where $d = g\alpha + f\beta + c = \frac{\Delta}{D}$

Now rotating the axes through an angle θ such that $\tan 2\theta = \frac{2h}{a-b}$ or, $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$, the equation (4)

reduces to $Ax^2 + By^2 + d = 0 \quad \ldots (5)$

(term in xy removed)

From (5), we get the canonical form.

Case 2 Equation (1) is a non-central conic (Parabola)

Here $\Delta \neq 0$, $D = 0$

As $D = ab - h^2 = 0$, the second degree ~~terms~~ terms in equation (1) will form a perfect square.

So, the equation (1) can be written as

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$$

or, $(\alpha x + \beta y)^2 = -2gx - 2fy - c \quad \ldots (6)$

Introducing a real number λ in (6), we write (6) as

$$(\alpha x + \beta y)^2 + 2\lambda(\alpha x + \beta y) + \lambda^2 = 2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + \lambda^2 - c$$

or, $(\alpha x + \beta y + \lambda)^2 = 2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + \lambda^2 - c \quad \ldots (7)$

We choose λ such that the two straight lines

$$\alpha x + \beta y + \lambda = 0 \quad \ldots (8)$$

and $2(\alpha\lambda - g) + 2(\beta\lambda - f)y + \lambda^2 - c = 0 \quad \ldots (9)$

are perpendicular.

So, $\alpha(\alpha\lambda - g) + \beta(\beta\lambda - f) = 0$

Or, $\lambda = \frac{\alpha g + \beta f}{\alpha^2+\beta^2}$

For this value of λ, (8) and (9) are perpendicular. The straight lines (8) and (9) are respectively the axis and the tangent at the vertex of the parabola for this value of λ. Solving (8) and (9) we get the vertex of the parabola.

Let us now choose these perpendicular straight lines as coordinate axes with reference to which a point $P(x,y)$ on the curve is (X, Y).

Then X = perpendicular distance of $P(x,y)$ from the straight line (9)

$$= \frac{2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + \lambda^2 - c}{2\sqrt{(\alpha\lambda-g)^2 + (\beta\lambda - f)^2}} \quad \cdots (10)$$

and Y = the perpendicular distance of $P(x,y)$ from the straight line (8)

$$= \frac{\alpha x + \beta y + \lambda}{\sqrt{\alpha^2+\beta^2}} \quad \cdots (11)$$

Thus, with reference to the new axes, equation (7) can be written as $(\alpha^2+\beta^2)Y^2 = 2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}\, X$

or, $$Y^2 = \frac{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}{\alpha^2+\beta^2} X \quad \ldots (12)$$

which is the canonical form of a parabola

whose axis is $Y=0$, i.e., $\alpha x+\beta y+\lambda=0$

and the tangent at the vertex is Y axis, i.e., $X=0$

or, $2(\alpha\lambda-g)+2(\beta\lambda-f)y+\lambda^2-c=0$

The length of the latus rectum is

$$\frac{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}{\alpha^2+\beta^2} = \frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{3/2}}$$

by putting the value of λ.

Worked Examples

1. Reduce the following equation to canonical form and hence determine the nature of the conic:

$$3x^2+2xy+3y^2-16x+20=0$$

Solution: Comparing the equation

$$3x^2+2xy+3y^2-16x+20=0 \quad \ldots (1)$$

with $ax^2+2hxy+by^2+2gx+2fy+c=0$, we have

$a=3,\ b=3,\ h=1,\ g=-8,\ f=0$ and $c=20$

So, $$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc+2fgh-af^2-bg^2-ch^2$$
$$= 3.3.20+2.0.(-8).1-3.0^2-3(-8)^2-20.1^2$$
$$= -32 \neq 0$$