and $D = ab - h^2 = 3.3 - 1^2 = 8 \neq 0$

So, equation (1) is central conic. Let the centre be (α, β). Then $3\alpha + \beta - 8 = 0$ ... (2)

and $\alpha + 3\beta = 0$ ... (3)

Solving (2) and (3), $\alpha = 3, \beta = -1$

So translating the origin to the centre $(3, -1)$, the equation (1) becomes (with (x', y') as the new coordinates) $3x'^2 + 2x'y' + 3y'^2 + d = 0$ (As terms in x and y vanish)

where $d = (-8) \times 3 + 0.(-1) + 20 = -4$

So, the equation can be written as

$$3x'^2 + 2x'y' + 3y'^2 - 4 = 0 \quad \ldots (4)$$

To remove the $x'y'$ term, we rotate the axes through an angle θ. Due to rotation, the equation (2) reduces to (with new coordinates (X, Y))

$$3(X\cos\theta - Y\sin\theta)^2 + 2(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 3(X\sin\theta + Y\cos\theta)^2 - 4 = 0$$

or, $(3 + \sin 2\theta)X^2 + 2\cos 2\theta XY + (3 - \sin 2\theta)Y^2 = 4$ ... (5)

For removal of XY term, we require that

$\cos 2\theta = 0$, that is $\theta = \frac{\pi}{4}$

Then (3) becomes $(3+1)X^2 + (3-1)Y^2 = 4$

or, $\dfrac{X^2}{1}+\dfrac{Y^2}{2}=1 \quad \cdots \text{(6)}$

This is the reduced canonical form of the given equation.

This represents an ellipse.

2. Reduce the following equation to canonical form and hence determine the nature of the conic:

$$x^2+4xy+y^2-2x+2y+6=0$$

Solution: Comparing the equation

$$x^2+4xy+y^2-2x+2y+6=0 \quad \cdots (1)$$

with the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, we have

$a=b=1$, $h=2$, $g=-1$, $f=1$ and $c=6$

So, $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & 6 \end{vmatrix} = -24 \neq 0$

and $D = ab-h^2 = -3 \neq 0$

Hence it is a central conic. Let the centre be (α,β).

Then $\alpha+2\beta-1=0 \quad \cdots (2)$

and $2\alpha+\beta+1=0 \quad \cdots (3)$

Solving (2) and (3), $\alpha=-1$, $\beta=1$

Let $d = g\alpha+f\beta+c = (-1)(-1)+1.1+6 = 8$

So, the reduced equation with centre as origin, is

$$x'^2 + 4x'y' + y'^2 + 8 = 0 \quad \cdots (4)$$

If, by rotation of axes, (4) is reduced to

$$AX^2 + BY^2 + 8 = 0,$$ then

by theory of invariants

$$A+B = 1+1 \quad \text{and} \quad AB = 1.1 - 2^2 = -3$$

So, $A = -1, 3$ and corresponding $B = 3, -1$

Hence the reduced equation to the canonical form of the given equation (1) is

either $-X^2 + 3Y^2 + 8 = 0$, i.e., $\dfrac{X^2}{8} - \dfrac{Y^2}{8/3} = 1$

or, $3X^2 - Y^2 + 8 = 0$, i.e., $\dfrac{Y^2}{8} - \dfrac{X^2}{8/3} = 1$,

each of which is a hyperbola

Note: The transformation can also be effected as in example 1 by the rotation of axes.

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3. Reduce the following equation to canonical form:

$$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$$

Solution: Here $a = 6,\ b = -6,\ h = -\frac{5}{2},\ g = 7;\ f = \frac{5}{2}$ and $c = 4$

Here $\Delta = 0$ and $D = -36 - \frac{25}{4} = -\frac{167}{4} \neq 0$

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Thus the given equation represents a pair of intersecting straight lines.

The equations giving their point of intersection (α,β),

is $a\alpha + h\beta + g = 0$

and $h\alpha + b\beta + f = 0$

So, $12\alpha - 5\beta + 14 = 0$

$-5\alpha - 12\beta + 5 = 0$

So, $\alpha = -\frac{11}{13}$ and $\beta = \frac{10}{13}$

Shifting the origin to the point (α,β) without changing the direction of axes, the given equation reduces to

$$6x'^2 - 5x'y' - 6y'^2 + d = 0$$

where $d = 7 \times \left(-\frac{11}{13}\right) + \frac{5}{2} \times \frac{10}{13} + 4 = 0$

or, $6x'^2 - 5x'y' - 6y'^2 = 0 \quad \ldots (1)$

Rotating the axes through an angle θ, (1) reduces to

$$6(X\cos\theta - Y\sin\theta)^2 - 5(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) - 6(X\sin\theta + Y\cos\theta)^2 = 0$$

or, $\left(6\cos 2\theta - \frac{5}{2}\sin 2\theta\right)X^2 - \left(5\cos 2\theta + 12\sin 2\theta\right)XY$
$$-\left(6\cos 2\theta - \frac{5}{2}\sin 2\theta\right)Y^2 = 0$$

To remove XY term, we put $5\cos\theta + 12\sin 2\theta = 0$

or, $\frac{\sin 2\theta}{-5} = \frac{\cos 2\theta}{12} = \frac{1}{13}$. With these