2.2 To find the condition that a given line may touch the conic

Let the equation of the conic and the line be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \cdots (1)$$

and

$$lx+my+n=0 \quad \cdots (2)$$

respectively.

From (2), $y=-\dfrac{lx+n}{m}$

Putting this value of y in (1), we have

$$x^2(am^2-2hlm+bl^2)-2x(hmn-bln-gm^2+flm)$$
$$+bn^2-2fmn+cm^2=0 \quad \cdots (3)$$

If the line (2) be a tangent, the roots of the equation (3) must be equal.

So, $(hmn-bln-gm^2+flm)^2-(am^2-2hlm+bl^2)(bn^2-2fmn+cm^2)=0$

or, $(bc-f^2)l^2+(ca-g^2)m^2+(ab-h^2)n^2+2(gh-af)mn$
$+2(hf-bg)nl+2(fg-ch)lm=0 \quad \cdots (4)$

It is the required condition.

Deductions:

(1) Circle: $x^2+y^2=a^2$

Condition: $a^2(l^2+m^2)-n^2=0$, i.e., $n=\pm a\sqrt{l^2+m^2}$

(2) Parabola: $y^2=4ax$

condition: $am^2-ln=0$, i.e., $n=\dfrac{am^2}{l}$

(3) Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Condition: $a^2l^2+b^2m^2-n^2=0$, i.e., $n=\pm\sqrt{a^2l^2+b^2m^2}$

(4) Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Condition: $a^2l^2-b^2m^2=n^2$ or $n=\pm\sqrt{a^2l^2-b^2m^2}$

Note: $y=mx+c$ will touch either (1), (2), (3) or (4)

if $c=\pm\sqrt{1+m^2}$, $c=\frac{a}{m}$, $c=\pm\sqrt{a^2m^2+b^2}$

or $c=\pm\sqrt{a^2m^2-b^2}$ respectively.

2.3 To find the equation of the normal to

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{at } (x_1,y_1)$$

Definition: The normal to a curve is the straight line perpendicular to the tangent at the point of contact.

The equation of the tangent at (x_1,y_1) is

$$axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0$$

or, $(ax_1+hy_1+g)x+(hx_1+by_1+f)y+gx_1+fy_1+c=0 \quad \ldots (1)$

Since the normal is perpendicular to (1) and passes through the point (x_1,y_1), its equation is

$$y-y_1=\frac{hx_1+by_1+f}{ax_1+hy_1+g}(x-x_1) \quad \ldots (2)$$

2.31 Particular Cases: (1) Parabola: $y^2=4ax$

The equation of the tangent at (x_1, y_1) is

$$yy_1 = 2a(x+x_1)$$

or, $$y = \frac{2a}{y_1}(x+x_1)$$

So, the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x-x_1)$$

Corollary: If we put $-\frac{y_1}{2a} = m$, then $x_1 = \frac{y_1^2}{4a} = am^2$

Thus the equation of the normal

$$y = mx - 2am - am^3$$

(1a) Show that three normals can be drawn to a parabola from a given point and the sum of the ~~other~~ ordinates of the feet of the normals is zero.

~~Soln~~: $(at^2, 2at)$ is a point ~~of~~ on the parabola $y^2=4ax$.

The equation of the normal at this point is

$$y - 2at = -\frac{2at}{2a}(x - at^2)$$

$$\text{or, } y + tx = 2at + at^3 \quad \ldots (1)$$

If this normal passes through a fixed point (h, k) then

$$k + th = 2at + at^3$$

$$\text{or } at^3 + (2a-h)t - k = 0 \quad \ldots (2)$$

So; (2) is a cubic equation in t. So, it has three roots. Corresponding to each of these three roots we have, on substitution, the equation of a normal passing through (h,k). Hence in general three normals can be drawn to a parabola through a given point.

Let t_1, t_2, t_3 be the roots of the equation (2).

From the relation between roots and coefficients,

$t_1 + t_2 + t_3 = 0$. If y_1, y_2, y_3 be the ordinates of the feet of normals, then

$y_1+y_2+y_3 = 2a(t_1+t_2+t_3) = 0$. Hence the result follows.

(2) Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

or, $$y = -\frac{b^2}{a^2}\frac{x_1}{y_1}x + \frac{b^2}{y_1}$$

So, the equation of the normal at (x_1, y_1) is

$$y-y_1 = \frac{a^2}{b^2}\frac{y_1}{x_1}(x-x_1), \text{ or, } \frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{y_1/b^2} \quad \cdots (1)$$

(2a) Show that four normals can be drawn to an ellipse through a given point and the