feet of the normals lie on a rectangular hyperbola.

Let the normal (1) passes through a given point $(h, k)$

Then $$\frac{h-x_1}{x_1/a^2} = \frac{k-y_1}{y_1/b^2} = \lambda \text{ (say)}$$

or, $\frac{x_1}{a} = \frac{ah}{a^2+\lambda}$, $\frac{y_1}{b} = \frac{bk}{b^2+\lambda}$

Since $(x_1, y_1)$ is on the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\text{or, } \frac{a^2h^2}{(a^2+\lambda)^2} + \frac{b^2k^2}{(b^2+\lambda)^2} = 1$$

It is a biquadratic equation in $\lambda$ and gives four values of $\lambda$. These four values correspond to four points on the ellipse and the normals at these points pass through the given point.

Again $a^2+\lambda = \frac{a^2h}{x_1}$, $b^2+\lambda = \frac{b^2k}{y_1}$

Subtracting $a^2-b^2 = \frac{a^2h}{x_1} - \frac{b^2k}{y_1}$. The point $(x_1, y_1)$ is the foot of a normal. Hence the locus of the feet of the normals is

$$a^2-b^2 = \frac{a^2h}{x} - \frac{b^2k}{y}$$

$$\text{or, } (a^2-b^2)xy = a^2hy - b^2kx$$

which is a rectangular hyperbola.

c) Hyperbola: $x^2/a^2 - y^2/b^2 = 1$

The equation of tangent at $(x_1, y_1)$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \text{or,} \quad y = \frac{b^2}{a^2}\frac{x_1}{y_1}x - \frac{b^2}{y_1}$$

So, the equation of the normal at $(x_1, y_1)$ is

$$y - y_1 = -\frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1)$$

or, $$\frac{x - x_1}{x_1/a^2} = -\frac{y - y_1}{y_1/b^2} \quad \cdots \quad (1)$$

9(a) Show that four normals can be drawn to a hyperbola through a given point and the locus of the feet of these normals is a rectangular hyperbola.

Let the normal (1) passes through a fixed point $(h, k)$.

Then $$\frac{h - x_1}{x_1/a^2} = -\frac{k - y_1}{y_1/b^2} = \lambda \text{ (say)}, \quad \frac{x_1}{a} = \frac{ah}{a^2+\lambda}, \quad \frac{y_1}{b} = \frac{bk}{b^2-\lambda}$$

We have $$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

So, $$\frac{a^2h^2}{(a^2+\lambda)^2} - \frac{b^2k^2}{(b^2-\lambda)^2} = 1$$

It is a biquadratic equation in $\lambda$ and gives four values of it. These four values correspond to four points on the hyperbola at which normals pass through the fixed point $(h, k)$

Again $a^2+\lambda = \frac{a^2h}{x_1}$, $b^2-\lambda = \frac{b^2k}{y_1}$

Adding $a^2+b^2 = \frac{a^2h}{x_1} + \frac{b^2k}{y_1}$

$(x_1, y_1)$ is a foot of one of the four normals.

Hence the locus of the feet of the normals

$$a^2+b^2 = \frac{a^2h}{x} + \frac{b^2k}{y} \quad \text{or,} \ (a^2+b^2)xy = a^2hy + b^2kx.$$

It is a rectangular hyperbola.

Worked Examples

1. Find the equation of the tangent and the normal at $(1, -1)$ to the conic $y^2-xy-2x^2-5y+x-6=0$.

Solution: The point $(1,-1)$ is on the conic. So, the equation of the tangent at $(1,-1)$ is

$$y.(-1) - \frac{1}{2}\left(x.(-1)+y.1\right) - 2x.1 - \frac{5}{2}(y-1) + \frac{1}{2}(x+1) - 6 = 0$$

or, $x+4y+3=0$

Since the normal is perpendicular to the tangent at $(1,-1)$, its equation is

$$y+1 = 4(x-1) \quad \text{or,} \ 4x-y-5=0$$

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2. Find the equations to the tangents to the conic $x^2+4xy+3y^2-5x-6y+3=0$ which are parallel to $x+4y=0$

Solution: Let the equation of any such tangent be

$x+4y+c=0$, or $x=-(4y+c)$.

Putting this value of x in the equation of the conic, we have

$(4y+c)^2-4y(4y+c)+3y^2+5(4y+c)-6y+3=0$

or, $3y^2+2(2c+7)y+c^2+5c+3=0$

Since the line is tangent to the conic, the above equation in y must have equal roots.

So, $4(2c+7)^2-4.3(c^2+5c+3)=0$

or, $c^2+13c+40=0$,

or, $(c+5)(c+8)=0$, $c=-5,-8$

So, the required tangents are $x+4y-5=0$, and $x+4y-8=0$.

3. Prove that the normal chord of a parabola at the point whose ordinate is equal to its abscissa, subtends a right angle at focus.

Solution Let $y^2=4ax$ ... (1)

be the equation of the parabola. The point whose ordinate is equal to abscissa, is obtained by