3. If $y = \frac{1}{x-\alpha}$ then $y_n = \frac{(-1)^n n!}{(x-\alpha)^{n+1}}$

4. If $y = \frac{1}{ax+b}$ then $y_n = \frac{(-1)^n n!}{(ax+b)^{n+1}} \cdot a^n$

5. If $y = e^{ax}$ then $y_n = a^n e^{ax}$

6. If $y = \sin ax$ then $y_n = a^n \sin\left(ax + \frac{n\pi}{2}\right)$

7. If $y = \cos ax$ then $y_n = a^n \cos\left(ax + \frac{n\pi}{2}\right)$

8. If $y = e^{ax}\cos(bx+c)$ $(a>0)$, then $y_n = r^n e^{ax}\cos(bx+c+n\phi)$,

where $r = \sqrt{a^2+b^2}$, $a = r\cos\phi$, $b = r\sin\phi$, $-\pi/2 < \phi < \pi/2$

9. If $y = e^{ax}\sin(bx+c)$ $(a>0)$, then $y_n = r^n e^{ax}\sin(bx+c+n\phi)$,

where $r = \sqrt{a^2+b^2}$, $a = r\cos\phi$, $b = r\sin\phi$; $-\pi/2 < \phi < \pi/2$

We give below proofs of 6 and 8 and leave others as exercise for the student:

6. If $y = \sin ax$, then $y_n = a^n \sin\left(ax + \frac{n\pi}{2}\right)$

Proof: $y_1 = a\cos ax = a\sin(ax + \pi/2)$, so the formula is true for $n=1$.

Assume that the formula is true for $n=m$. So, $y_m = a^m \sin\left(ax + \frac{m\pi}{2}\right)$

Now, $y_{m+1} = a^{m+1}\cos\left(ax + \frac{m\pi}{2}\right) = a^{m+1}\sin\left(ax + \frac{m\pi}{2} + \pi/2\right)$

$= a^{m+1}\sin\left(ax + (m+1)\pi/2\right)$. So, the formula is true for $n = m+1$. So, by principle of mathematical induction, $y_n = a^n \sin(ax + n\pi/2)$ for all positive integer n.

8. If $y = e^{ax}\cos(bx+c)$ $(a>0)$ then $y_n = r^n e^{ax}\cos(bx+c+n\phi)$
where $r = \sqrt{a^2+b^2}$, $a = r\cos\phi$, $b = r\sin\phi$, $-\pi/2 < \phi < \pi/2$

Proof: $y_1 = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c)$
$= e^{ax}\left[a\cos(bx+c) - b\sin(bx+c)\right]$
$= e^{ax}r\left[\cos\phi\cos(bx+c) - \sin\phi\sin(bx+c)\right]$, putting
$a = r\cos\phi$, $b = r\sin\phi$. So, $r = \sqrt{a^2+b^2}$ and
$-\pi/2 < \phi < \pi/2$ as $a>0$.

So, $y_1 = e^{ax}r\cos(bx+c+\phi)$. So, the formula is true for $n=1$.

Let the formula be true for $n = m$. So,
$y_m = r^m e^{ax}\cos(bx+c+m\phi)$

then $y_{m+1} = r^m a e^{ax}\cos(bx+c+m\phi) - r^m b e^{ax}\sin(bx+c+m\phi)$
$= r^m e^{ax}\left[a\cos(bx+c+m\phi) - b\sin(bx+c+m\phi)\right]$
$= r^{m+1}e^{ax}\left[\cos(bx+c+m\phi)\cos\phi - \sin(bx+c+m\phi)\sin\phi\right]$
putting $a = r\cos\phi$, $b = r\sin\phi$

$= r^{m+1}e^{ax}\cos\{bx+c+(m+1)\phi\}$

So, the formula is true for $n = m+1$. So, principle of mathematical induction,
$y_n = r^n e^{ax}\cos(bx+c+n\phi)$ for all positive integer n
where $r = \sqrt{a^2+b^2}$ and $-\pi/2 < \phi < \pi/2$

2.1 Leibnitz's Theorem on successive derivatives

If u and v be two functions of x, both derivable at least upto n times, then $y = uv$ is derivable n times and the nth derivative of $y = y_n = (uv)_n$ is given by the formula

$$(uv)_n = \sum_{r=0}^{n} {}^nC_r\, u_{n-r} v_r = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v + \cdots + {}^nC_n u v_n \quad (\text{taking } u_0 = u,\ v_0 = v)$$

$$= u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + u v_n$$

Proof: $(uv)_1 = u_1 v + u v_1$. So, the theorem is true for $n = 1$

Let us assume that the theorem is true for $n = m$.

So, $(uv)_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \cdots + {}^mC_r u_{m-r} v_r + \cdots + {}^mC_m u v_m$

Differentiating both sides once more, we have

$$(uv)_{m+1} = \{u_{m+1} v + u_m v_1\} + {}^mC_1 \{u_m v_1 + u_{m-1} v_2\} + \cdots + {}^mC_r \{u_{m-r+1} v_r + u_{m-r} v_{r+1}\} + {}^mC_m \{u_1 v_m + u v_{m+1}\}$$

$$= u_{m+1} v + (1 + {}^mC_1) u_m v_1 + ({}^mC_1 + {}^mC_2) u_{m-1} v_2 + \cdots + ({}^mC_{r-1} + {}^mC_r) u_{m-r+1} v_r + \cdots + ({}^mC_{m-1} + {}^mC_m) u_1 v_m + u v_{m+1}$$

$$= u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \cdots + {}^{m+1}C_r u_{m-r} v_r + \cdots + {}^{m+1}C_m u_1 v_m + u v_{m+1}$$

$$\left(\text{using the result: } {}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r\right)$$

So, the Theorem is true for $n = m+1$. Hence by principle of mathematical induction, the theorem is true for all positive integer n.

Worked ~~exp~~ examples

1. If $f(x) = \frac{x^3}{x^2-1}$, prove that for $n>1$, $f^n(0) = 0$ if n be even
$= -n!$ if n be odd.

Solution: $f(x) = x + \frac{x}{x^2-1} = x + \frac{1}{2}\left[\frac{1}{x+1} + \frac{1}{x-1}\right]$

So, $f^n(x) = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+1)^{n+1}} + \frac{(-1)^n n!}{(x-1)^{n+1}}\right]$ for $n>1$

That is, for $n>1$, $f^n(0) = \frac{1}{2}(-1)^n n!\left[1 + (-1)^{n+1}\right]$

So, for $n>1$, $f^n(0) = 0$ if n be even
$= -n!$ if n be odd

2. If $y = \frac{1}{x^2+a^2}$, prove that $y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$, where
$\cot\theta = \frac{x}{a}$.

Solution: Let $x = a\cot\theta$. Then $\frac{dx}{d\theta} = -a\,\mathrm{cosec}^2\theta$, $y = \frac{1}{a^2}\sin^2\theta$

So, $y_1 = \frac{1}{a^2}\sin 2\theta \cdot \frac{d\theta}{dx} = \frac{1}{a^2}\sin 2\theta\left(-\frac{\sin^2\theta}{a}\right) = -\frac{1}{a^3}\sin^2\theta \sin 2\theta$

So, the formula is true for $n=1$. Assume that the formula is true for $n=m$. That is, $y_m = \frac{(-1)^m m!}{a^{m+2}}\sin^{m+1}\theta \sin(m+1)\theta$

Now $y_{m+1} = \frac{(-1)^m m!}{a^{m+2}}\left[(m+1)\sin^m\theta\cos\theta\sin(m+1)\theta + (m+1)\sin^{m+1}\theta\cos(m+1)\theta\right]\frac{d\theta}{dx}$

or, $y_{m+1} = \frac{(-1)^m (m+1)!}{a^{m+2}}\sin^m\theta \sin\left((m+1)\theta+\theta\right) \times \left(-\frac{\sin^2\theta}{a}\right)$

or, $y_{m+1} = \frac{(-1)^{m+1}(m+1)!}{a^{m+3}}\sin^{m+2}\theta \sin(m+2)\theta$