or, $(1+x^2)y_2 + xy_1 - m^2 y = 0 \quad \ldots \quad (1)$

Differentiating (1) n times and using Leibnitz's Theorem, we get

$$(1+x^2)y_{n+2} + n y_{n+1}(2x) + \frac{n(n-1)}{2} y_n \cancel{(2)} \times 2 + x y_{n+1} + n y_n - m^2 y_n = 0$$

or, $(1+x^2) y_{n+2} + (2n+1) x y_{n+1} + (n^2 - m^2) y_n = 0$

At $x=0$, $y_{n+2}(0) = (m^2 - n^2) y_n(0) \quad \ldots \quad (2)$

But $y(0) = 1$, $y_1(0) = m$, $y_2(0) = m^2$. Using (2), we have

$y_3(0) = (m^2 - 1^2) y_1(0) = m(m^2 - 1^2)$

$y_4(0) = (m^2 - 2^2) y_2(0) = m^2 (m^2 - 2^2)$

So, we conclude,

$y_n(0) = m(m^2 - 1^2)(m^2 - 3^2) \cdots (m^2 - (n-2)^2)$, if n be odd

$= m^2 (m^2 - 2^2)(m^2 - 4^2) \cdots (m^2 - (n-2)^2)$, if n be even.

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8. If $y = (x^2 - 1)^n$, then prove that

$(x^2 - 1) y_{n+2} + 2x y_{n+1} - n(n+1) y_n = 0$. Hence show that

if $z = \frac{d^n}{dx^n}(x^2 - 1)^n$, then z satisfies the following second

order differential equation (Known as Legendre's Equation)

$$(1-x^2)\frac{d^2 z}{dx^2} - 2x \frac{dz}{dx} + n(n+1) z = 0$$

Solution: $y = (x^2 - 1)^n$. So, $y_1 = n(x^2 - 1)^{n-1} \cdot (2x) = \dfrac{2nx (x^2 - 1)^n}{x^2 - 1}$

or, $(x^2 - 1) y_1 = 2nxy \quad \ldots \quad (1)$

Differentiating the equation (1) (n+1) times and using Leibnitz's theorem, we have,

$$\left\{ (x^2-1) y_1 \right\}_{n+1} = 2n\,(xy)_{n+1}$$

$$\text{or,}\quad (x^2-1)\,y_{n+2} + (n+1)\,y_{n+1}\,(2x) + \frac{n(n+1)}{2!}\,y_n \times 2 = 2n\left[x y_{n+1} + (n+1) y_n\right]$$

$$\text{or,}\quad (x^2-1)\,y_{n+2} + 2x\,y_{n+1} - n(n+1)\,y_n = 0 \qquad \text{— (2)}$$

The relation (2) can be written as

$$\left[(1-x^2)\,y_{n+2} - 2x\,y_{n+1}\right] + n(n+1)\,y_n = 0$$

$$\text{or,}\quad \frac{d}{dx}\left\{(1-x^2)\,y_{n+1}\right\} + n(n+1)\,y_n = 0$$

$$\text{or,}\quad \frac{d}{dx}\left\{(1-x^2)\,\frac{dz}{dx}\right\} + n(n+1)\,y_n = 0 \quad \left(\because\ z = \frac{d^n}{dx^n}(x^2-1)^n = y_n\right)$$

So, z satisfies the equation

$$(1-x^2)\,\frac{d^2 z}{dx^2} - 2x\,\frac{dz}{dx} + n(n+1)\,z = 0$$

~~Example~~ 9. Use Leibnitz's theorem to find the nth derivatives of (i) $e^{ax+b}\sin x$ (ii) $(ax+b)^n \cos x$

Solution (i) $\left\{e^{ax+b}\sin x\right\}_n = (uv)_n$ where $u = e^{ax+b}$, $v = \sin x$

Now for any positive integer k, $u_k = a^k e^{ax+b}$ and

$v_k = \sin\left(k\pi/2 + x\right)$

So, by Leibnitz's theorem, we have

$$\{e^{ax+b}\sin x\}_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \dots + uv_n$$

$$= a^n e^{ax+b}\sin x + n a^{n-1}e^{ax+b}\cos x - \frac{n(n-1)}{2!}a^{n-2}e^{ax+b}\sin x + \dots + e^{ax+b}\sin\left(\frac{n\pi}{2}+x\right).$$

(ii) $\{(ax+b)^n\cos x\}_n = (uv)_n$ where $u = (ax+b)^n$, $v = \cos x$

Now for any positive integer k, $u_k = n(n-1)\cdots(n-k+1)a^k(ax+b)^{n-k}$

and $v_k = \cos(k\pi/2 + x)$

So, by Leibnitz's theorem, we have

$$\{(ax+b)^n\cos x\}_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \dots + uv_n$$

$$= \cancel{n!\,a^n\cos x + n\times n!\,a^{n-1}\cos(\pi/2+x) + \frac{n(n-1)}{2!}\cdot\frac{n!}{2!}a^{n-2}}$$

$$= n!\,a^n\cos x + n\times n!\,a^{n-1}(ax+b)\cos(\pi/2+x) + \frac{n(n-1)}{2!}\cdot\frac{n!}{2!}a^{n-2}(ax+b)^2\cos(2\cdot\pi/2+x) + \dots + (ax+b)^n\cos\left(n\frac{\pi}{2}+x\right)$$

$$= n!\,a^n\cos x - n\times n!\,a^{n-1}(ax+b)\sin x - \frac{n(n-1)\cdot n!}{(2!)^2}a^{n-2}(ax+b)^2\cos x + \dots + (ax+b)^n\cos\left(\frac{n\pi}{2}+x\right)$$

Exercises

1. If $y = \tan^{-1}\frac{1+x}{1-x}$, prove that $y_n = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\cot\theta = x$.

2. If $y = x^n\log x$, show that $y_n = (n-1)! + n\,y_{n-1}$.
Also prove that $y_n = n!\left[\log x + 1 + \frac{1}{2} + \dots + \frac{1}{n}\right]$

3. Use Leibnitz's theorem to find the nth derivatives of
(i) $e^{ax+b}\cos x$ (ii) $(ax+b)^n\sin x$

4. If $y = \tan^{-1}x$, prove that

(i) $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$ and

(ii) $y_n(0) = 0$, if n be even

$= (-1)^{\frac{1}{2}(n-1)}(n-1)!$, if n be odd.

5. If $y = \cos(m\sin^{-1}x)$, prove that

(i) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$ and

(ii) $y_n(0) = 0$, if n be odd

$= -m^2(2^2-m^2)(4^2-m^2)\cdots\{(n-2)^2-m^2\}$, if n be even

6. If $y = e^{a\sin^{-1}x}$, prove that

(i) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$ and

(ii) $y_n(0) = \{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\cdots(2^2+a^2)a^2$, if n be even

$= \{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\cdots(1^2+a^2)a$, if n be odd

7. If $y = a\cos(\log x) + b\sin(\log x)$, prove that

$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

8. If $y = \log(x+\sqrt{1+x^2})$, prove that

$y_{2n}(0) = 0$ and $y_{2n+1}(0) = (-1)^n 1^2.3^2.5^2\cdots(2n-1)^2$.

9. If $y = e^{-x}x^n$, prove that

$xy_{n+2} + (x+1)y_{n+1} + (n+1)y_n = 0$

deduce that $x\frac{d^2z}{dx^2} + (1-x)\frac{dz}{dx} + nz = 0$, where

$z = \frac{1}{n!}e^x\frac{d^n}{dx^n}(e^{-x}x^n)$.