3. L'Hospital's rule

First we discuss about indeterminate forms. If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m \neq 0$, we know that $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{l}{m}$. If, however, $m = 0$, then the limit could not be evaluated. The case when $l = 0$, $m = 0$ was not covered earlier. In this case, the limit of the quotient $\frac{f}{g}$ is said to have the indeterminate form $\frac{0}{0}$.

The other indeterminate forms are represented by the symbols $\frac{\infty}{\infty}$, $\infty - \infty$, 0^0, 1^{∞}, $1^{-\infty}$, ∞^0.

L'Hospital's rule is to determine the limit of these indeterminate forms.

We state some theorems related to L'Hospital's rule without proof.

Theorem 3.1 (L'Hospital's Rule for $0/0$ form) If f, g are two functions such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) $f'(x)$, $g'(x)$ exists and $g'(x) \neq 0$, $\forall x \in (a-\delta, a+\delta)$, $\delta > 0$ except possibly at a and $\left[f'(x) = \frac{d}{dx}(f(x)),\ g'(x) = \frac{d}{dx}(g(x))\right]$

(iii) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Theorem 3.2 (Generalised L'Hospital's Rule for $0/0$ form)

If f, g are two functions and such that

(i) $f^n(x)$, $g^n(x)$ exist and $g^n(x) \neq 0$ $(n = 1, 2, \ldots, n)$ $\left[f^n(x) = \frac{d^n}{dx^n}(f(x)),\ g^n(x) = \frac{d^n}{dx^n}(g(x))\right]$ for any x in $(a-\delta, a+\delta)$, $\delta > 0$ except possibly at $x = a$

(ii) $\lim_{x\to a} f(x) = \lim_{x\to a} f'(x) = \cdots = \lim_{x\to a} f^{n-1}(x) = 0$ and

$\lim_{x\to a} g(x) = \lim_{x\to a} g'(x) = \dots = \lim_{x\to a} g^{n-1}(x)$ and

(iii) $\lim_{x\to a} \frac{f^n(x)}{g^n(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^n(x)}{g^n(x)}$

Indeterminate form $\frac{\infty}{\infty}$

If $f(x)$ and $g(x)$ both tend to ∞ as $x\to a$ then $\lim_{x\to a} \frac{f(x)}{g(x)}$ takes ∞/∞ form.

Theorem 3.3 (L'Hospital's rule for ∞/∞ form) If f, g be two functions such that (i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$,

(ii) $f'(x), g'(x)$ exist and $g'(x) \neq 0$, $\forall x$ in $(a-\delta, a+\delta)$, $\delta>0$ except possibly at a, and

(iii) $\lim_{x\to a} \frac{f'(x)}{g'(x)} = l$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = l$

Indeterminate form $0\times\infty$

When $f(x)\to 0$ and $g(x)\to\infty$ as $x\to a$, $f(x).g(x)$ takes $0\times\infty$ form. However $f(x).g(x)$ may be expressed as

$\frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$ which has respectively $\frac{0}{0}$ and $\frac{\infty}{\infty}$ form.

Indeterminate form $\infty - \infty$

This can be reduced to the form $0/0$ or ∞/∞, for

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}$$

Indeterminate form $0^0, 1^\infty, \infty^0$

This form can be made to depend upon the previous form by putting $z = (f(x))^{g(x)}$, so that $\log z = g(x).\log(f(x))$.

Then it is of the form $0/0$ or ∞/∞. So, we find $\lim \log z$

and as $\lim \log z = \log \lim z$ and we get $\lim z$.

Now using L'Hospital's rule, we solve some problems.

1. Evaluate the limit : $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$, using L'Hospital's rule

Solution: Here the limit takes $0/0$ form. We use L'Hospital's rule successively. The evaluation of the can be exhibited as follows:

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - \frac{2}{1+x}}{x\cos x + \sin x} \quad \left(\frac{0}{0} \text{ form}\right) \quad (\text{using L'Hospital's rule})$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + \frac{2}{(1+x)^2}}{-x\sin x + 2\cos x} = 1 \quad (\text{using L'Hospital's rule})$$

2. Evaluate $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x^3}$ using L'Hospital's rule.

Solution: $$\lim_{x \to 0} \frac{x - \tan x}{x^3} \quad \left(0/0 \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2} \quad \left(\frac{0}{0} \text{ form}\right) \quad (\text{using L'Hospital's rule})$$

$$= \lim_{x \to 0} \frac{-2\sec^2 x \tan x}{6x}$$

$$= -\frac{1}{3} \lim_{x \to 0} \frac{\sec^2 x \tan x}{x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= -\frac{1}{3}\lim_{x\to 0}\left(\sec^4 x + \tan x \times 2\sec^2 x \tan x\right) \quad \text{(using L'Hospital's rule)}$$

$$= -\frac{1}{3}$$

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3. Evaluate $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

solution: $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$ ($0/0$ form)

$$= \lim_{x\to 0} \frac{xe^x + e^x - \frac{1}{1+x}}{2x} \quad (0/0 \text{ form}) \quad \text{(using L'Hospital's rule)}$$

$$= \lim_{x\to 0} \frac{xe^x + 2e^x + \frac{1}{(1+x)^2}}{2} \quad \text{(using L'Hospital's rule)}$$

$$= \frac{3}{2}$$

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4. Evaluate $\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$

Solution: $\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$ $\left(\frac{0}{0} \text{ form as } \lim_{x\to 0}(1+x)^{1/x} = e\right)$

$$= \lim_{x\to 0} \frac{(1+x)^{1/x}\{x - (1+x)\log(1+x)\}}{x^2(1+x)} \quad \text{(using L'Hospital's rule)}$$

$$= e.\lim_{x\to 0} \frac{x - (1+x)\log(1+x)}{x^2(1+x)} \quad (0/0 \text{ form})$$

$$= e.\lim_{x\to 0} \frac{-\log(1+x)}{2x + 3x^2} \quad (0/0 \text{ form}) \quad \text{(using L'Hospital's rule)}$$