$$= e \cdot \lim_{x\to 0} \frac{-1}{(2+6x)(1+x)} = -\frac{e}{2}$$ (using L'Hospital's rule)

5. Find $\lim_{x\to 1-} \frac{\log(1-x)}{\cot(\pi x)}$

Solution It is of $\frac{\infty}{\infty}$ form and so,

$$\lim_{x\to 1-} \frac{\log(1-x)}{\cot(\pi x)} = \lim_{x\to 1-} \frac{-\frac{1}{1-x}}{-\pi \operatorname{cosec}^2(\pi x)} = \lim_{x\to 1-} \frac{\sin^2 \pi x}{\pi(1-x)} \quad [0/0 \text{ form}]$$

$$= \lim_{x\to 1-} \frac{\pi \sin(2\pi x)}{-\pi} = 0$$ (using L'Hospital's rule)

6. Evaluate $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

Solution: It is a $\infty - \infty$ form. We therefore write it in $0/0$ form as follows:

$$\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right) = \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \quad [0/0 \text{ form}]$$

$$= \lim_{x\to 0} \frac{\sin 2x - 2x}{2x \sin^2 x + x^2 \sin 2x} \quad [0/0 \text{ form}]$$ (using L'Hospital's rule)

$$= \lim_{x\to 0} \frac{2\cos 2x - 2}{2\sin^2 x + 2x\sin 2x + 2x \sin 2x + 2x^2 \cos 2x}$$

$$= \lim_{x\to 0} \frac{\cos 2x - 1}{\sin^2 x + 2x \sin 2x + x^2 \cos x} \quad [0/0 \text{ form}]$$

$$= \lim_{x\to 0} \frac{-2\sin 2x}{3\sin 2x + 6x\cos 2x - 2x^2 \sin 2x} \quad [0/0 \text{ form}]$$ (using L'Hospital's rule)

$$= \lim_{x\to 0} \frac{-4\cos 2x}{6\cos 2x + 6\cos 2x - 12x \sin 2x - 4x \sin 2x - 4x^2 \cos 2x} = \frac{-4}{12} = -\frac{1}{3}$$ (using L'Hospital's rule)

7. Evaluate $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

Solution : It is a 1^∞ form. So,

let $y=\left(\frac{\tan x}{x}\right)^{1/x^2}$. So, $\log y=\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$

So, $\lim\limits_{x\to 0}\log y=\lim\limits_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}$ ($0/0$ form)

$=\lim\limits_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}-\frac{1}{x}}{2x}$ (using L'Hospital's rule)

$=\lim\limits_{x\to 0}\frac{x\sec^2 x-\tan x}{2x^2\tan x}$ [$0/0$ form]

$=\lim\limits_{x\to 0}\frac{\sec^2 x+2x\sec^2 x\tan x-\sec^2 x}{4x\tan x+2x^2\sec^2 x}$ (using L'Hospital's rule)

$=\lim\limits_{x\to 0}\frac{\sec^2 x\tan x}{2\tan x+x\sec^2 x}$

$=\lim\limits_{x\to 0}\frac{\tan x}{\sin 2x+x}$ [$0/0$ form]

$=\lim\limits_{x\to 0}\frac{\sec^2 x}{2\cos 2x+1}$ [using L'Hospital's rule]

$=\frac{1}{3}$

So, $\lim\limits_{x\to 0}\log y=\frac{1}{3}$ or, $\log\lim\limits_{x\to 0}y=\frac{1}{3}$

or, $\lim\limits_{x\to 0}y=e^{1/3}$. So, $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}=e^{1/3}$

8. Find the values of p and q such that

$$\lim_{x\to 0} \frac{x(1-p\cos x)+q\sin x}{x^3} = \frac{1}{3}$$

Solution: The given ratio has the indeterminate form $0/0$.

Using L'Hospital's rule, the limit should be

$$= \lim_{x\to 0} \frac{1-p\cos x + xp\sin x + q\cos x}{3x^2} \quad \text{——} \; (1)$$

Since the denominator $\to 0$ as $x\to 0$, therefore, in order that the limit of (1) be finite as $x\to 0$, it is necessary that the numerator should also $\to 0$ as $x\to 0$. This gives

$$1-p+q=0 \quad \text{——} \; (2)$$

In this case the limit should be of of the form $0/0$.

So, using L'Hospital's rule again, limit in (1)

$$= \lim_{x\to 0} \frac{(p-q)\sin x + p\sin x + xp\cos x}{6x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{(p-q)\cos x + p\cos x + p\cos x - px\sin x}{6} \quad (\text{using L'Hospital's rule})$$

$$= \frac{3p-q}{6} = \frac{1}{3} \; (\text{given})$$

This gives $\quad 3p-q=2 \quad$ —— (3)

Solving p and q from (2) and (3), we get $p=\frac{1}{2}$, $q=-\frac{1}{2}$

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9. Find a and b in order that

$$\lim_{x\to 0} \frac{a\sin 2x - b\sin x}{x^3} = 1$$

Solution: It is of the form $0/0$. So, using L'Hospital's rule this limit $= \lim_{x\to 0} \frac{2a\cos 2x - b\cos x}{3x^2}$, if it exists. ...(1)

Since denominator $\to 0$ as $x \to 0$, in order that limit of (1) exists finitely, Numerator should tend to zero as $x \to 0$

So, $2a - b = 0$ .... (2)

So, limit of (1) $= \lim_{x\to 0} \frac{b\cos 2x - b\cos x}{3x^2}$ $\left(\frac{0}{0} \text{ form}\right)$ (from (2))

So, using L'Hospital's rule this limit $= \lim_{x\to 0} \frac{-2b\sin 2x + b\sin x}{6x}$ $\left(\frac{0}{0}\text{ form}\right)$

Using L'Hospital's rule once again, we obtain the limit as

$$\lim_{x\to 0} \frac{-4b\cos 2x + b\cos x}{6} = -\frac{3b}{6} = 1 \text{ (given)}$$

So, $b = -2$

So, from (2), $a = -1$

So, $a = -1$, $b = -2$

<u>Exercises</u> Using L'Hospital's rule evaluate the following limits:

1. $\lim_{x\to 0} \frac{x - \sin x}{x^3}$

2. $\lim_{x\to 0} \frac{x - \log(1+x)}{1 - \cos x}$

3. $\lim_{x\to 0} \frac{\log(1+x^3)}{\sin^3 x}$

4. $\lim_{x\to \infty} \frac{x^3}{e^x}$

5. $\lim_{x\to \pi/2+} \frac{\log(x - \pi/2)}{\tan x}$

6. $\lim_{x\to\infty} x\tan\frac{1}{x}$

7. $\lim_{x\to 0} \left(\cot^2 x - \frac{1}{x^2}\right)$

8. $\lim_{x\to \pi/2} (\sec x - \tan x)$