9. $\lim\limits_{x \to 0+} (\cot x)^{\sin x}$

10. $\lim\limits_{x \to \pi/2} (\sin x)^{\tan x}$

11. $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)^{1/x}$

12. $\lim\limits_{x \to 0} \dfrac{e^{x} - 2\cos x + e^{-x}}{x \sin x}$

13. $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

14. $\lim\limits_{x \to 1} x^{\frac{1}{x-1}}$

15. Find the values of a and b in order that

$\lim\limits_{x \to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}$ may be equal to 1

16. Find the value of a and the limit in order that $\lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$ be finite.

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7. Reduction formula

If an integral is dependent on the positive integer n, then we sometimes find a recurrence formula, i.e., a relation between the original integral and those ~~integral some integra~~ integrals, with n reduced. For example, ~~I~~ if I_n be the original integral, n is a positive integer, then

$$I_n = 3I_{n-1} + 2I_{n-2}$$ is a reduction formula for I_n

It will be clear from the examples that follows:

<u>Example 1</u> Find a reduction formula for $\int \sin^n x \, dx$, n is a positive integer > 1

Solution: Let $I_n = \int \sin^n x\,dx = \int \sin^{n-1} x \cdot \sin x\,dx$

$= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)\,dx$ [Integrating by parts]

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$

$= -\sin^{n-1} \cos x + (n-1) I_{n-2} - (n-1) I_n$

So, $I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

or, $n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

or, $I_n = \dfrac{-\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$. This is the required reduction formula.

Example 2 Find a reduction formula for $\int \cos^n x\,dx$, n being a positive integer >1. Hence evaluate $\int_0^{\pi/2} \cos^n x\,dx$

Solution Let $I_n = \int \cos^n x\,dx$

Then $I_n = \int \cos^{n-1} x \cdot \cos x\,dx$

$= \cos^{n-1} x \cdot \sin x - \int (n-1)\cos^{n-2} x(-\sin x)\sin x\,dx$ [Integrating by parts]

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$

So, $I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$

or, $I_n + (n-1) I_n = \cos^{n-1}x \sin x + (n-1) I_{n-2}$

So, $I_n = \dfrac{\cos^{n-1}x \sin x}{n} + \dfrac{n-1}{n} I_{n-2} \cdots (1)$. This is the required reduction formula.

Let $J_n = \int_0^{\pi/2} \cos^n x \, dx$

So, $J_n = \left[\dfrac{\cos^{n-1}x \sin x}{n} \right]_0^{\pi/2} + \dfrac{n-1}{n} J_{n-2}$ [from (1)]

So, $J_n = \dfrac{n-1}{n} J_{n-2}$

Therefore, $J_n = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \dfrac{2}{3} J_1, & \text{when } n \text{ is odd} \\ \dfrac{n-1}{n} \dfrac{n-3}{n-2} \cdots \dfrac{3}{4} J_2, & \text{when } n \text{ is even} \end{cases}$

As $J_1 = \int_0^{\pi/2} \cos x \, dx = 1$ and $J_2 = \int_0^{\pi/2} \cos^2 x \, dx = \frac{1}{2} \int_0^{\pi/2} (1+\cos 2x) \, dx$
$= \frac{1}{2} \times \frac{\pi}{2}$

So, $J_n = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \dfrac{2}{3} & \text{when } n \text{ is odd} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \pi/2 & \text{when } n \text{ is even} \end{cases}$

Note: $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$ $\left[\text{as } \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \right]$

Example 3 Find a reduction formula for $\int \tan^n x \, dx$ where n is a positive integer > 1.

Solution

Let $I_n = \int \tan^n x \, dx$

So, $I_n = \int \tan^{n-2} x \cdot \tan^2 x \, dx = \int \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int \tan^{n-2}x \sec^2 x\,dx - \int \tan^{n-2}x\,dx$

$= \frac{\tan^{n-1}x}{n-1} - I_{n-2}$

So, $I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$ which is the required reduction formula.

Example 4 Find a reduction formula for $\int \sec^n x\,dx$, n is a positive integer >1.

Solution: Let $I_n = \sec^n x\,dx$. Then

$I_n = \int \sec^{n-2}x \cdot \sec^2 x\,dx$

$= \sec^{n-2}x \cdot \tan x - \int (n-2)\sec^{n-3}x \sec x \tan x \cdot \tan x\,dx$

(Integrating by parts)

$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x \tan^2 x\,dx$

$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x(\sec^2 x - 1)\,dx$

$= \sec^{n-2}x \tan x - (n-2) I_n + (n-2) I_{n-2}$

So, $(n-1)I_n = \sec^{n-2}x \tan x + (n-2) I_{n-2}$

Hence $I_n = \frac{\sec^{n-2}x \tan x}{(n-1)} + \frac{n-2}{n-1} I_{n-2}$, which is the required reduction formula.

Example 5 If $I_{m,n} = \int \sin^m x \cos^n x\,dx$ (m, n are both positive integers) then find a reduction formula for $I_{m,n}$