$m > 1, n > 1$

Solution: Let $I_{m,n} = \int \sin^m \cos^n dx$

Then $I_{m,n} = \int \sin^m x \cos^n x \, dx$

$$= \int \cos^{n-1} x \left(\sin^m x \cos x \right) dx$$

$$= \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} - \int (n-1) \cos^{n-2} x \left(-\sin x\right) \frac{\sin^{m+1} x}{m+1} dx \quad \text{(Integrating by parts)}$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \left(1 - \cos^2 x\right) dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \, dx - \frac{n-1}{m+1} \int \cos^n x \sin^m x \, dx$$

Thus, $I_{m,n} = \dfrac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \dfrac{n-1}{m+1} I_{m,n-2} - \dfrac{n-1}{m+1} I_{m,n}$

So, $\left(1 + \dfrac{n-1}{m+1}\right) I_{m,n} = \dfrac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \dfrac{n-1}{m+1} I_{m,n-2}$

or, $I_{m,n} = \dfrac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \dfrac{n-1}{m+n} I_{m,n-2}$

Similarly, we can show that

$$I_{m,n} = -\frac{\cos^{n+1} x \sin^{m-1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

So, $I_{m,n} = \dfrac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \dfrac{n-1}{m+n} I_{m,n-2}$

$$= -\frac{\cos^{n+1} x \sin^{m-1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \quad \text{are}$$

the required reduction formula

Note: you can deduce any one of them only.

Example 7 Obtain a reduction formula for $\int \frac{\sin^m x}{\cos^n x}\,dx$, where m, n are both positive integers greater than 1.

Solution : Let $I_{m,n} = \int \frac{\sin^m x}{\cos^n x}\,dx$

Now $\frac{d}{dx}\left(\frac{\sin^m x}{\cos^n x}\right) = \frac{\cos^n x\left(m \sin^{m-1} x \cos x\right) - \sin^m x \left(n \cos^{n-1} x\left(-\sin x\right)\right)}{\left(\cos^n x\right)^2}$

$= m \frac{\sin^{m-1} x}{\cos^{n-1} x} + n \frac{\sin^{m+1} x}{\cos^{n+1} x} \quad \ldots (1)$

Integrating, we get

$$\frac{\sin^m x}{\cos^n x} = m\, I_{m-1,\, n-1} + n\, I_{m+1,\, n+1}$$

Therefore replacing m by $m-1$ and n by $n-1$, we get

$$\frac{\sin^{m-1} x}{\cos^{n-1} x} = (m-1) I_{m-2,\, n-2} + (n-1)\, I_{m,n}$$

Hence $I_{m,n} = \frac{1}{n-1} \frac{\sin^{m-1} x}{\cos^{n-1} x} - \frac{m-1}{n-1} I_{m-2,\, n-2}$

Example 8 Obtain a reduction formula for $\int \frac{dx}{(x^2+a^2)^n}$, where n is a positive integer ≥ 1 and $a \in \mathbb{R}$, $\mathbb{R}$ is the set of all real numbers and $a \neq 0$.

Solution: Let $I_n = \int \frac{dx}{(x^2+a^2)^n}$

~~Solution~~ We have,

$$I_n = \frac{1}{(x^2+a^2)^n}\, x - \int \frac{(-n)}{(x^2+a^2)^{n+1}} \cdot 2x \cdot x \cdot dx \quad \text{(Integrating by parts)}$$

$$= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{\{(x^2+a^2) - a^2\}}{(x^2+a^2)^{n+1}}\, dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{dx}{(x^2+a^2)^n} - 2a^2 n \int \frac{dx}{(x^2+a^2)^{n+1}}$$

Replacing n by $n-1$, we have the reduction formula

$$I_{n-1} = \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\, I_{n-1} - 2a^2(n-1)\, I_n$$

or, $$2a^2(n-1)\, I_n = (2n-3)\, I_{n-1} + \frac{x}{(x^2+a^2)^{n-1}}$$

Hence $$I_n = \frac{2n-3}{2a^2(n-1)}\, I_{n-1} + \frac{1}{2a^2(n-1)} \cdot \frac{x}{(x^2+a^2)^{n-1}}$$

Example 9 If m be a positive integer, prove that

$$\int_0^{\pi/2} \cos^m x \sin mx\, dx = \frac{1}{2^{m+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^m}{m}\right]$$

Solution : Let $I_{m,m} = \int_0^{\pi/2} \cos^m x \sin mx \, dx$

Then $I_{m,m} = \left[\cos^m x \left(-\frac{\cos mx}{m}\right)\right]_0^{\pi/2} - \int_0^{\pi/2} m \cos^{m-1} x (-\sin x)\left(-\frac{\cos mx}{m}\right) dx$

(Integrating by parts)

$= \frac{1}{m} - \int_0^{\pi/2} \cos^{m-1} x (\cos mx \sin x) \, dx$

$= \frac{1}{m} - \int_0^{\pi/2} \cos^{m-1} x \cdot \left[\sin mx \cos x - \sin (m-1) x \, dx\right] dx$

$= \frac{1}{m} - \left[\int_0^{\pi/2} \cos^{m-1} x \sin mx \cos x \, dx - \int_0^{\pi/2} \cos^{m-1} x \sin (m-1) x \, dx\right]$

$= \frac{1}{m} - \int_0^{\pi/2} \cos^m x \sin mx \, dx - \int_0^{\pi/2} \cos^{m-1} x \sin (m-1) x \, dx$

$= \frac{1}{m} - I_{m,m} - I_{m-1,m-1}$

So, $2 I_{m,m} = \frac{1}{m} + I_{m-1,m-1}$ or, $I_{m,m} = \frac{1}{2m} + \frac{1}{2} I_{m-1,m-1}$

Now $I_{m-1,m-1} = \frac{1}{2(m-1)} + \frac{1}{2} I_{m-2,m-2}$ and $I_{m-2,m-2} = \frac{1}{2(m-2)} + \frac{1}{2} I_{m-3,m-3}$

and so on.

So, $I_{m,m} = \frac{1}{2m} + \frac{1}{2}\left[\frac{1}{2(m-1)} + \frac{1}{2} I_{m-2,m-2}\right]$

$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2} I_{m-2,m-2}$

$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \frac{1}{2^3} I_{m-3,m-3}$

$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \dots + \frac{1}{2^{m-1}} \cdot I_{1,1}$