$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \cdots + \frac{1}{2^{m-1}}\cdot\frac{1}{2} \quad \left[\text{as } I_{1,1} = \tfrac{1}{2}\right]$$

$$= \frac{1}{2^{m+n}}\left[\frac{2^{m+n}}{2m} + \frac{2^{m+n}}{2^2(m-1)} + \frac{2^{m+n}}{2^3(m-2)} + \cdots \frac{2^{m+n}}{2^m}\right]$$

$$= \frac{1}{2^{m+n}}\left[\frac{2}{1} + \frac{2^2}{2} + \cdots + \frac{2^{m-1}}{m-1} + \frac{2^m}{m}\right] \quad \text{(Proved)}$$

Example 10 If $I_n = \int_0^{\pi/2} x^n \sin x \, dx$, n being a positive integer $n > 1$, show that
$$I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}.$$
Hence find the value $\int_0^{\pi/2} x^5 \sin x \, dx$

Solution: $I_n = \int_0^{\pi/2} x^n \sin x \, dx$

$$= \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} n x^{n-1}(-\cos x)\, dx$$

$$= 0 + n\int_0^{\pi/2} x^{n-1}\cos x \, dx.$$

$$= n\left[\left[x^{n-1}\sin x\right]_0^{\pi/2} - \int_0^{\pi/2}(n-1)x^{n-2}\sin x \, dx\right]$$

$$= n\left[\left(\pi/2\right)^{n-1} - (n-1)\int_0^{\pi/2} x^{n-2}\sin x \, dx\right]$$

So, $I_n + n(n-1) I_{n-2} = n\left(\pi/2\right)^{n-1}$ (proved)

Putting $n=5$, we have $I_5 + 5(5-1) I_3 = 5(\pi/2)^4$

or, $I_5 = 5(\pi/2)^4 - 20 I_3$

Again putting $n=3$ in the above reduction formula,

we have, $I_3 + 6 I_1 = 3(\pi/2)^2$

So, $I_3 = 3(\pi/2)^2 - 6\int_0^{\pi/2} x \sin x \, dx$

$= 3(\pi/2)^2 - 6\left[\left[x(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2}(-\cos x)\,dx\right]$

$= 3(\pi/2)^2 - 6\left[0 + \int_0^{\pi/2}\cos x\,dx\right]$

$= 3(\pi/2)^2 - 6\left[-\cos x\right]_0^{\pi/2} = 3(\pi/2)^2 - 6$

So, $I_5 = 5(\pi/2)^4 - 20\left[3(\pi/2)^2 - 6\right]$

$= 5(\pi/2)^4 - 60(\pi/2)^2 + 120$

i.e., $\int_0^{\pi/2} x^5 \sin x \, dx = 5(\pi/2)^4 - 60(\pi/2)^2 + 120$

Example 11 Let $I_n = \int_0^1 x^n \tan^{-1} x \, dx \; (n > 1)$. Prove that,

$(n+1) I_n + (n-1) I_{n-2} = \pi/2 - \frac{1}{n}$, n is an integer.

Solution: Now $I_n = \int_0^1 x^n \tan^{-1} x \, dx$

$$= \left[\tan^{-1}x \, \frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{1}{1+x^2} \frac{x^{n+1}}{n+1}$$

$$= \frac{\pi}{4} \cdot \frac{1}{n+1} - \frac{1}{n+1} \int_0^1 \frac{x^{n+1}}{1+x^2} dx$$

So, $(n+1) I_n = \frac{\pi}{4} - \int_0^1 \frac{x^2}{1+x^2} \cdot x^{n-1} dx$

$$= \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{1+x^2}\right) x^{n-1} dx$$

$$= \frac{\pi}{4} - \int_0^1 x^{n-1} dx + \int_0^1 \frac{x^{n-1}}{1+x^2} dx$$

$$= \pi/4 - \left[\frac{x^n}{n}\right]_0^1 + \left[x^{n-1} \tan^{-1}x\right]_0^1 - (n-1) \int_0^1 x^{n-2} \tan^{-1}x \, dx$$

$$= \pi/4 - \frac{1}{n} + \pi/4 - (n-1) \int_0^1 x^{n-2} \tan^{-1}x \, dx$$

$$= \frac{\pi}{2} - \frac{1}{n} - (n-1) I_{n-2}$$

Hence $(n+1) I_n + (n-1) I_{n-2} = \pi/2 - \frac{1}{n}$ (Proved)

Example 12 Let $I_n = \int_0^\pi \frac{1-\cos nx}{1-\cos x} dx$, where n is a positive integer or zero. Prove that $I_{n+2} + I_n = 2 I_{n+1}$

Solution: Now $I_n + I_{n-2} = \int_0^\pi \frac{1-\cos nx}{1-\cos x} dx + \int_0^\pi \frac{1-\cos(n+2)x}{1-\cos x} dx$

$$= \int_0^\pi \frac{1-\cos nx + 1 - \cos(n+2)x}{1-\cos x} dx = \int_0^\pi \frac{2 - \{\cos nx + \cos(n+2)x\}}{1-\cos x} dx$$

$$= 2\int_0^{\pi} \frac{2 - 2\cos(n+1)x\cos x}{1-\cos x}\,dx = 2\int_0^{\pi} \frac{1-\cos(n+1)x\cos x}{1-\cos x}\,dx$$

$$= 2\int_0^{\pi} \frac{1-\cos(n+1)x + \cos(n+1)x - \cos(n+1)x\cos x}{1-\cos x}\,dx$$

$$= 2\int_0^{\pi} \frac{1-\cos(n+1)x}{1-\cos x} + 2\int_0^{\pi} \cos(n+1)x\,dx$$

$$= 2\,I_{n+1} + 2\left[\frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi}$$

$$= 2\,I_{n+1} + 0 \quad = 2\,I_{n+1}$$

So, $I_{n+2} + I_n = 2\,I_{n+1}$ (Proved)

Exercises: 1. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\,dx$ prove that $I_{m,n} = \frac{m}{m+n} I_{m-1,\,n-1}$ and deduce that $I_{m,n} = \frac{\pi}{2^{m+1}}$, m, n are positive integers.

2. Let $J_{m,n} = \int_0^{\pi/2} \cos mx \cos nx\,dx$, m, n are positive integers. Prove that $(m^2 - n^2) J_{m,n} + m(m-1) J_{m-2,n} = 0$

3. If $I_n = \int_0^{a} (a^2 - x^2)^n\,dx$, n is a positive integer, prove that $I_n = \frac{2na^2}{2n+1} I_{n-1}$