Solution: Since p and q are primes > 3, p and q are of the form $3k+1$ or $3k+2$ where k is an integer

If both p and q are either of the forms $3k+1$ or $3k+2$, then $3 \mid (p-q)$. If one of p and q is of the form $3k+1$ and the other is of the form $3k+2$, then $3 \mid (p+q)$

So, in any case, $3 \mid (p^2-q^2)$.

Since p and q are odd primes, p and q are of the form $4k+1$ or $4k+2$ where k is an integer

If both p and q are of the form $4k+1$, then $2 \mid (p+q)$ and $4 \mid (p-q)$.

If both p and q are of the form $4k+3$, then $2 \mid (p+q)$ and $4 \mid p-q$.

If one of p and q is of the form $4k+1$ and the other is $4k+3$, Then $4 \mid (p+q)$ and $2 \mid (p-q)$

So, in any case $8 \mid (p^2-q^2)$

Since 3 and 8 are prime to each other,

$$3.8 = 24 \mid (p^2-q^2)$$

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4. If p and p^2+8 are both prime numbers, prove that $p=3$.

Solution: Any integer p is one of the forms $3k$, $3k+1$, $3k+2$ where k is an integer

If $p = 3k+1$, then $p^2+8 = (3k+1)^2+8 = 9k^2+6k+9 = 3(3k^2+2k+3)$.

Since p^2+8 is a prime $3k^2+2k+3$ must be 1

for some integer k and in that ~~p~~ case p^2+8 must be 3.
But for no integer k, $3k^2+2k+3$ can be 1 and for no integer k, p^2+8 can be 3. So, $p = 3k+1$ is an impossibility.

If $p = 3k+2$, then $p^2+8 = (3k+2)^2+8 = 3(3k^2+4k+4)$.

~~Since~~ Since p^2+8 is a prime, $3k^2+4k+4$ must be 1 for some integer k and in that case p^2+8 must be 3.

By similar arguments, $p = 3k+2$ is an impossibility.

So, $p = 3k$, where k is an integer. Since p is a prime, k must be 1 and so, $p = 3$.

5. If 2^n-1 be a prime, prove that n is a prime

Solution: Let n be composite. Then $n = p.q$ where p and q are integers > 1.

$$2^n-1 = 2^{pq}-1 = (2^p)^q-1 = (2^p-1)\left(2^{p(q-1)}+2^{p(q-2)}+\cdots+2^p+1\right)$$

Each fact~~or~~ in the right is greater than 1 and therefore 2^n-1 is ~~composition~~ composite, a contradiction to the fact that 2^n-1 is prime.

So, our assumption is wrong. So, n is prime.

6. Prove that n^4+4^n is a composite number for all $n>1$

Solution: ~~Cat~~ Case I Let n be even

Then $n^4 + 4^n$ is divisible by 4 and so it is a composite number

<u>Case 2</u> Let n be odd and $n = 2k+1$, where k is a ~~integer~~ natural number.

Then $n^4 + 4^n = n^4 + 4 \cdot 4^{2k} = n^4 + 4a^4$ where $a = 2^k$

$= (n^2 + 2a^2)^2 - (2an)^2 = (n^2 + 2an + 2a^2)(n^2 - 2an + 2a^2)$

$n^2 + 2an + 2a^2 = (n+a)^2 + a^2$ and $n^2 - 2an + 2a^2 = (n-a)^2 + a^2$

So, $n^2 + 2an + 2a^2 > 1$ and $n^2 - 2an + 2a^2 > 1$ as

a is a positive integer > 1

Consequently, $n^4 + 4^n$ is a composite number for all $n > 1$

7. Let p be a prime and a be a positive integer. Prove that a^n is divisible by p if and only if a is divisible by p.

Solution: Let a be divisible by p. Then $a = pk$ for some integer k.

$a^n = p^n k^n = p(p^{n-1} k^n) = pm$ where m is an integer.

This shows that a^n is divisible by p ... (i)

Since p is a prime, let a be not divisible by p.

$\gcd(a, p) = 1$. So, there exists integers u and v such that $au + pv = 1$,

Then $a^n u^n = (1 - pv)^n = 1 - ps$ where s is an integer.

or, $a^n r + ps = 1$ where r, s are integers.

This shows that $\gcd(a^n, p) = 1$ and therefore

a^n is not divisible by p

Hence a is not divisible by $p \Rightarrow a^n$ is not divisible by p

Contrapositively, $p \mid a^n \Rightarrow p \mid a$ .... (ii)

From (i) and (ii) the desired result is obtained.

Note: If p and q are two statements then contrapositive statement of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$ where $\sim p$ mean negation of p. Also $p \Rightarrow q$ is true if and only if $\sim q \Rightarrow \sim p$ is true.

For example, let p be the statement it rains and q be the statement the sky is cloudy,

So, $p \Rightarrow q$ is true then $\sim q \Rightarrow \sim p$ is true

i.e., As the sky is not cloudy, it is not raining.

Theorem 2.4.6 (Euclid's theorem) The number of primes is infinite.

Proof: we prove the theorem by contradiction.

Let us suppose that the number of primes be finite and let p be the greatest prime.

we write the primes $2, 3, 5, 7, \ldots$ in succession and p is the last in the enumeration.

The product $2.3.5.7.\ldots.p$ in which every prime appears only once is divisible by each prime and therefore the number $(2.3.5.7\ldots p)+1$ is not divisible