Theorem 1.2.2: Let R be an equivalence relation on a non-empty set S and $a, b \in S$. If $a \not R b$ then $cl(a)$ and $cl(b)$ are disjoint.

Proof: If possible, let $cl(a) \cap cl(b) \neq \phi$ and $p \in cl(a) \cap cl(b)$

So, $p \in cl(a)$ and $p \in cl(b)$. Hence pRa and pRb.

This implies aRp and pRb as R is symmetric.

This implies aRb, since R is transitive.

This is a contradiction. So, our assumption is false.

Hence $cl(a) \cap cl(b) = \phi$.

Note: The Relation R is sometimes denoted by the roman letter ρ (rho).

Theorem 1.2.3 Let R be an equivalence relation on a non-empty set S and $a, b \in S$. Then the classes $cl(a)$ and $cl(b)$ are either equal or disjoint.

Proof: Since $a, b \in S$ and R is a relation on S, either aRb or $a \not R b$.

Let aRb. Then $cl(a) = cl(b)$ by Theorem 1.2.1

Let $a \not R b$, Then $cl(a)$ and $cl(b)$ are disjoint, by Theorem 1.2.2.

Consequently, either $cl(a) = cl(b)$ or $cl(a) \cap cl(b) = \phi$.

Example: 2. Find the equivalence classes determined by the equivalence relation R on $\mathbb{Z}$ by aRb if and only if $a-b$ is divisible by 5 for $a, b \in \mathbb{Z}$.

There are five distinct equivalence classes. They are

$cl(0) = \{0, \pm 5, \pm 10, \pm 15, \ldots\} = \{5n : n \text{ is an integer}\}$

$cl(1) = \{1, 1\pm 5, 1\pm 10, \ldots\} = \{5n+1 : n \text{ is an integer}\}$

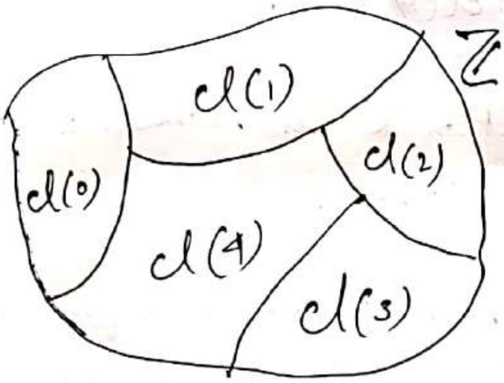
$cl(2) = \{2, 2\pm 5, 2\pm 10, \ldots\} = \{5n+2 : n \text{ is an integer}\}$

$cl(3) = \{3, 3\pm 5, 3\pm 10, \ldots\} = \{5n+3 : n \text{ is an integer}\}$

$cl(4) = \{4, 4\pm 5, 4\pm 10, \ldots\} = \{5n+4 : n \text{ is an integer}\}$

We see that $cl(0) \cup cl(1) \cup cl(2) \cup cl(3) \cup cl(4) = \mathbb{Z}$ and also any two of the classes are disjoint

Partition of $\mathbb{Z}$



Note 1: A family of ~~sets~~ non-empty sets is said to be a partition of a nonempty set S if S is the union of all the sets of the family and each pair of the family of ~~sets~~ sets are disjoint. So, if $\{S_\alpha : \alpha \in I\}$ be a family of sets, I is the index set then it is said to be a ~~family~~ partition of a non-empty set S if

(i) $\bigcup_{\alpha \in I} S_\alpha = S$.

and (ii) $S_\alpha \cap S_\beta = \phi$, for $\alpha, \beta \in I$ and $\alpha \neq \beta$.

So, we see that, in the above example, $\{cl(0), cl(1), cl(2), cl(3), cl(4)\}$ gives a partition of the set $\mathbb{Z}$

Note 2: Let $a \in \mathbb{Z}$. By division algorithm $\exists$ integers ($\exists$ means there exists) q and r such that $a = 5q + r$, $0 \le r < 5$, r is called the remainder or the least non-negative residue $a \pmod 5$. $a \in cl(r)$. For example $34 = 6 \times 5 + 4$

So, $35 \in cl(4)$. $-27 = 5 \times (-6) + 3$. So, $-27 \in cl(3)$

$25 = 5 \times 5 + 0$. So, $25 \in cl(0)$. $-15 = 5 \times 3 + 0$. So, $-15 \in cl(0)$.

So, we see when we divide an integer by 5, the least non-negative residues possible, are, 0, 1, 2, 3 and 4. Here also we see $cl(0), cl(1), cl(2), cl(3)$ and $cl(4)$ are the only five distinct equivalence classes. They are also ~~do~~ called the classes of residues of $\mathbb{Z} \pmod 5$

Note 3 As $0R5$, $5R10$, $10R(-5)$, etc in the above example, we have $cl(0) = cl(5) = cl(10) = cl(-5)$ etc.

As $0 \not R 1$, so $cl(0) \cap cl(1) = \phi$ etc.

Theorem 1.2.4 An equivalence relation R on a non-empty set S determines a partition of S. Conversely, each partition of S generates an equivalence relation on S.

Proof: ~~$\{cl(a) : a \in S\}$ determines a partition of S as for $a \in S$, $cl(a) \ne \phi$ as $a \in cl(a)$~~. So, $\{cl(a) : a \in S\}$ is a non-empty family of subsets of S and $\bigcup_{a \in S} cl(a) = S$ where $\{cl(a) : a \in S\}$ are the family of distinct equivalence classes generated by R. Also $cl(a) \cap cl(b) = \phi$. So, this family gives a partition of S.

Conversely, let $\{S_\alpha : \alpha \in I\}$ be a partition of S.
Let us define a relation R on S by a R b if a, b belong to the same S_α, for some $\alpha \in I$.
Let $a \in S$. Then aRa as a and a belong to the same subset of the partition. So R is reflexive.
Let $a, b \in S$ and aRb. Then $a, b \in S_\alpha$ for some $\alpha \in I$
So, $b, a \in S_\alpha$ for that α. Hence bRa.
So, R is symmetric.
Let $a, b, c \in S$ and aRb and bRc. Then $a, b \in S_\alpha$ for some $\alpha \in I$ and $b, c \in S_\beta$ for some $\beta \in I$.
Now S_α and S_β are subsets of a partition, must either be identical or disjoint. Since $b \in S_\alpha \cap S_\beta$, it follows that $S_\alpha = S_\beta$. So, $a, c \in S_\alpha$
Hence aRc. So R is transitive.
Hence R is an equivalence relation.

1.3. Partial order relation

Definition: Let S be a non-empty set. A relation R on the set S is said to be antisymmetric if aRb and $bRa \Rightarrow a = b$ for $a, b \in S$.

Examples: 1. The relation ρ defined on $\mathbb{R}$ (the set of all real numbers) by $x \rho y$ if and only if $x \leq y$ for $x, y \in \mathbb{R}$, is antisymmetric.