Worked Examples 1. Find the number of positive integers less than $n$ and prime to $n$, when $n = 324$ and $900$

Solution : $324 = 2^2 \cdot 3^4$. So, $\phi(324) = 324\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = 108$

Hence the number of positive integers less than 324 and prime to 324 is 108

$900 = 2^2 \cdot 3^2 \cdot 5^2$. So, $\phi(900) = 900\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = 240$

Hence the number of positive integers less than 900 and prime to 900 is 240.

2. If $n$ be an odd positive integer, prove that $\phi(2n) = \phi(n)$

Solution : Since $n$ is odd, $\gcd(2, n) = 1$

So, $\phi(2n) = \phi(2)\phi(n) = \phi(n)$, since $\phi(2) = 1$.

3. If $n$ be an even positive integer, prove that

$\phi(2n) = 2\phi(n)$

Solution : Let $n = 2^k \cdot p$ where $p$ is an odd positive integer. Then

$$\begin{aligned}\phi(n) &= \phi(2^k)\phi(p) \\ &= 2^k\left(1-\tfrac{1}{2}\right)\phi(p) \\ &= 2^{k-1}\phi(p)\end{aligned}$$

and

$$\begin{aligned}\phi(2n) = \phi(2^{k+1}p) = \phi(2^{k+1})\phi(p) &= 2^{k+1}\left(1-\tfrac{1}{2}\right)\phi(p) \\ &= 2^k\phi(p)\end{aligned}$$

So, $\phi(2n) = 2 \cdot 2^{k-1}\phi(p) = 2\phi(n)$.

We now define another arithmetic function, called $\tau(n)$

(pronounced as tau n)

Definition: The number of positive divisors of a positive integer $n$ is denoted by $\tau(n)$

If $n=1$, then $\tau(n)=1$

Let $n$ be a positive integer greater than 1. Then $n$ can be expressed as $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where the primes $p_1, p_2, \ldots, p_r$ are distinct with $p_1 < p_2 < \cdots < p_r$ and the exponents $\alpha_i$, $i=1,2,\ldots,r$ are all positive integers.

If $m$ be a positive divisor of $n$, then $m$ is of the form $p_1^{u_1} \cdot p_2^{u_2} \cdots p_r^{u_r}$ where $u_1, u_2, \ldots u_r$ are integers such that $0 \le u_i \le \alpha_i$, $i=1,2,\ldots,r$

Thus the positive divisors of $n$ are in one-to-one correspondence with the totality of $r$-tuples $(u_1, u_2, \ldots, u_r)$, where $0 \le u_i \le \alpha_i$, $i=1,2,\ldots,r$

The number of such $r$ tuples is $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)$

Hence the total number of positive divisors of $n$ is $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)$

So, $\tau(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)$

For example $\tau(48) = \tau(2^4 \cdot 3) = (4+1)(1+1) = 10$

<u>Theorem 4.6.3</u> $\tau(n)$ is odd if and only if $n$ is a perfect square

Proof: When $n=1$, it is a perfect square and $\tau(1)=1$ and it is odd. Let $n(>1)$ be a perfect square and let the canonical form of $n$ be $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes and $p_1 < p_2 < \cdots < p_r$ and the exponents $\alpha_i$, $i = 1, 2, \ldots, r$, are all positive integers.

Then each of $\alpha_1, \alpha_2, \ldots, \alpha_r$ is an even integer and

$\tau(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)$ is odd (product of odd integers).

Conversely, let $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)$ be odd. Then each of the factor $\alpha_1+1, \alpha_2+1, \ldots, \alpha_r+1$ must be odd. Consequently each of $\alpha_1, \alpha_2, \ldots, \alpha_r$ must be even and $n$ is therefore a perfect square. This completes the proof.

Worked Examples

1. Find $\tau(360)$ and $\tau(900)$

Solution: $360 = 2^3 \cdot 3^2 \cdot 5$. So, $\tau(360) = (1+3)(1+2)(1+1) = 24$

$900 = 2^2 \cdot 3^2 \cdot 5^2$. So, $\tau(900) = (1+2)(1+2)(1+2) = 27$

2. Find the number of odd positive divisors of 2700

Solution: $2700 = 2^2 \cdot 3^3 \cdot 5^2$, Every positive divisor of 2700 is of the form $2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3}$, where $0 \le \alpha_1 \le 2$, $0 \le \alpha_2 \le 3$, $0 \le \alpha_3 \le 2$.

Each term in the product $(1+2+2^2)(1+3+3^2+3^3)(1+5+5^2)$ is a positive divisor of 2700 and conversely.

The odd positive divisors of 2700 are given by the terms of the product $1 \cdot (1+3+3^2+3^3)(1+5+5^2)$.

So, the number of odd positive divisors of are $(3+1)(2+1) = 12$

3. If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ where $p_1, p_2, \ldots, p_r$ are distinct primes and ~~for~~ $p_1 < p_2 < \cdots < p_r$ and $\alpha_i$ are positive integers, $i = 1, 2, \ldots, r$. Prove that the number of positive square free divisors of $n$ is $2^k$.

Solution: A positive square free divisor of $n$ is of the form $p_1^{u_1} p_2^{u_2} \cdots p_r^{u_r}$, $0 \le u_i \le 1$, $i = 1, 2, \ldots, r$ and they are in one-to-one correspondence with the totality of $r$-tuples $(u_1, u_2, \ldots, u_r)$, where $0 \le u_i \le 1$, $i = 1, 2, \ldots, r$

The number of such $r$-tuples is $2^k$

Hence the number of positive squarefree divisors of $n$ is $2^k$

4. Find the smallest number having 8 positive divisors.

Solution: $8 = 2 \cdot 2 \cdot 2 = 2^2 \cdot 2 = 2^3$

Let $n$ be a number with 8 positive divisors

The factorisation $8 = 2 \cdot 2 \cdot 2$ indicates that $n$ is of the form $p_1 \cdot p_2 \cdot p_3$, where $p_1, p_2, p_3$ are distinct primes. For example, $n = 2 \cdot 3 \cdot 7$

The factorisation of $8 = 4 \cdot 2$ indicates that number $n$ is of the form $p_1^3 p_2$ where $p_1, p_2$ are distinct primes.

For example, $n = 3^3 \cdot 2$