The factorisation of $8 = 8 \cdot 1$ indicates that the number n is of the form $n = p_1^7$, where p_1 is a prime.

For example $n = 2^7$

Therefore the ~~number~~ number n is one of the forms

$p_1 \cdot p_2 \cdot p_3$, $p_1^3 \cdot p_2$, p_1^7 where p_1, p_2, p_3 are distinct primes

Clearly, the least number is $2^3 \cdot 3 = 24$ as the least number of the form $p_1 \cdot p_2 \cdot p_3$ is $2 \cdot 3 \cdot 5 = 30$, the least number of the form $p_1^3 \cdot p_2 = 2^3 \cdot 3 = 24$ and the least number of the form p_1^7 is $2^7 = 128$.

We define another arithmetic function $\sigma(n)$ (sigma n)

Definition: The sum of all positive divisors of a positive integer n is denoted by $\sigma(n)$.

If $n = 1$, 1 is the ~~not~~ only positive divisor, so, $\sigma(1) = 1$

Let n be a positive integer greater than 1. Then n can be expressed as $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where the primes p_i are distinct with $p_1 < p_2 < \cdots < p_r$ and α_i are positive integers, $i = 1, 2, \ldots, r$

Every positive divisor of n is a term of the product

$$(1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1})(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}) \cdots (1 + p_r + p_r^2 + \cdots + p_r^{\alpha_r})$$

$\cancel{\frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdots \frac{p_r^{\alpha_r+1}-1}{p_r-1}}$ and conversely, each term in the product is a divisor of n.

Hence the sum of all positive divisors of n

$$= \sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdots \frac{p_r^{\alpha_r+1}-1}{p_r-1}$$

So, $\sigma(1)=1$ and for $n>1$, $\sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha_2+1}-1}{p_2-1} \cdots \frac{p_r^{\alpha_r+1}-1}{p_r-1}$

Definition:

An arithmetic function f is said to be multiplicative if $f(mn) = f(m)f(n)$ for all positive integers m, n such that m, n are prime to each other.

Example: $\phi(n)$ is a multiplicative function.

Theorem 4.6.4 The functions τ and σ are both multiplicative functions.

Proof: Let m, n be relatively prime positive integers. $\tau(mn) = \tau(m)\tau(n)$ holds trivially if either m is 1 or n is 1.

We assume that $m>1$ and $n>1$

Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ and $n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}$

where $p_1, p_2, \ldots, p_r$ and $q_1, q_2, \ldots q_s$ are set of distinct ~~set~~ of primes. and α_i, β_j $i=1,2,\ldots r$ and $j=1,2,\ldots s$ are positive integers. Since m and n are relatively prime, each p_i is different from each q_j. So, the prime factorization of mn can be written as

$$mn = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \cdot q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}$$

So, $\tau(mn) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_r+1)(\beta_1+1)(\beta_2+1)\cdots(\beta_s+1)$

$= \tau(m)\tau(n)$

~~Again~~ $\sigma(mn) = \dfrac{p_1^{\alpha_1+1}-1}{p_1-1} \dfrac{p_2^{\alpha_2+1}-1}{p_2-1} \cdots \dfrac{p_r^{\alpha_r+1}-1}{p_r-1} \cdot \dfrac{q_1^{\beta_1+1}-1}{q_1-1} \cdot \dfrac{q_2^{\beta_2+1}-1}{q_2-1} \cdots \dfrac{q_s^{\beta_s+1}-1}{q_s-1}$.

$= \sigma(m)\sigma(n)$

Hence τ and σ are multiplicative functions.

Definition (Perfect number) A positive integer n is said to be a perfect number if $\sigma(n) = 2n$, i.e., if n be the sum of its positive divisors excluding itself.

For example, 6 is a perfect number; 28 is another.

Worked examples 1. Find $\sigma(360)$ and $\sigma(900)$

Solution: $360 = 2^3 \cdot 3^2 \cdot 5$. So, $\sigma(360) = \dfrac{2^4-1}{2-1} \cdot \dfrac{3^3-1}{3-1} \cdot \dfrac{5^2-1}{5-1}$

$= 15 \cdot 13 \cdot 6 = 1170$

$900 = 2^2 \cdot 3^2 \cdot 5^2$. So, $\sigma(900) = \dfrac{2^3-1}{2-1} \cdot \dfrac{3^3-1}{3-1} \cdot \dfrac{5^3-1}{5-1} = 7 \cdot 13 \cdot 31$

$= 2821$

2. Find the sum of all even positive divisors of 2700.

Solution: $2700 = 2^2 \cdot 3^3 \cdot 5^2$. Every positive divisor of 2700 is of the form $2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3}$, where $\alpha_1, \alpha_2, \alpha_3$ are integers such that ~~0 ≤ α₁ ≤ 2~~, ~~0≤~~ i.e. $0 \le \alpha_1 \le 2$, $0 \le \alpha_2 \le 3$, $0 \le \alpha_3 \le 2$

Therefore each term in the product

$(1+2+2^2)(1+3+3^2+3^3)(1+5+5^2)$ is a positive divisor of 2700 and conversely.

The even positive divisors of 2700 are given by different terms of the product $(2+2^2)(1+3+3^2+3^3)(1+5+5^2)$

So, the sum of even positive divisors

$$= (2+2^2)(1+3+3^2+3^3)(1+5+5^2) = 6 \cdot 40 \cdot 31 = 7440.$$

3. Let k be a positive integer and $k > 1$ and $2^k - 1$ is a prime. If $n = 2^{k-1}(2^k - 1)$ then show that n is a perfect number.

Solution: $2^k - 1$ is an odd prime, say p

$\sigma(n) = \sigma(2^{k-1} p) = \sigma(2^{k-1})\sigma(p)$, since 2^{k-1} and p are prime to each other.

$\sigma(2^{k-1}) = 1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$ and

$\sigma(p) = 1 + p$

So, $\sigma(n) = (2^k - 1)(1 + p) = (2^k - 1) 2^k$

$= 2n$

This proves that n is a perfect number.

4. If $d_1, d_2, \ldots, d_k$ be the ~~set~~ list of all positive divisors of a positive integer n, prove that $\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k} = \frac{\sigma(n)}{n}$.

Solution: d_i is a positive divisor $\Rightarrow \frac{n}{d_i}$ is also a positive divisor. As d_i runs through all positive divisors of n, $\frac{n}{d_i}$ also does so.

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