Now we cover the syllabus of Unit-3 of the syllabus which are:

, Rank of a matrix, inverse of a matrix, characterization of invertible matrices.

, System of linear equations, row reduction and echelon forms, vector equations, the matrix equation $AX = B$, solution sets of linear systems, applications of linear systems.

5. Inverse of a matrix :

Inverse of a matrix (Definition) : A square matrix A of order n is said to be invertible if there exits a matrix B such that $AB = BA = I_n$, I_n is the identity matrix of order n. B is said to an inverse of A.

In order that both AB and BA should exist, B must be a square matrix of order n.

Theorem 5.1 An invertible matrix has a unique inverse.

Proof: Let A be an invertible matrix of order n. If possible, let B and C be two inverses of A.

Then $AB = BA = I_n$, $AC = CA = I_n$

We have $C(AB) = (CA)B$, since multiplication is associative

$$\Rightarrow CI_n = I_n B$$

$$\Rightarrow C = B$$

This proves that A has a unique inverse.

Scanned with CamScanner

In view of this theorem, we can now speak of 'the inverse' of an invertible matrix. The unique inverse of A is denoted by A^{-1} and it satisfies the relation $AA^{-1} = A^{-1}A = I_n$

<u>Definition</u> A square matrix A is said to be non-singular if $\det A \neq 0$ ($\det A$ is the determinant of A, also denoted by $|A|$)

<u>Theorem 5.2</u> An $n \times n$ matrix A is invertible if and only if A is non-singular.

Proof: Let A be an $n \times n$ invertible matrix. Then there exists matrix B such that $AB = BA = I_n$.

We have $\det(AB) = \det I_n = 1$ or,

$$\det A \cdot \det B = 1 \quad (\because \det(AB) = \det A \cdot \det B)$$

This implies $\det A \neq 0$ and so, A is non-singular.

Conversely, let A be a non-singular matrix of order n.

Then $\det A \neq 0$

For an $n \times n$ matrix, we know that

$$A \cdot (\operatorname{adj} A) = (\operatorname{adj} A) \cdot A = (\det A) I_n$$

Since $\det A \neq 0$, $A \cdot \frac{1}{\det A}(\operatorname{adj} A) = \frac{1}{\det A}(\operatorname{adj} A) \cdot A = I_n$

From the definition of an inverse, it follows that $\frac{1}{\det A}(\operatorname{adj} A)$ is the inverse of A. Hence A is invertible.

Scanned with CamScanner

Note 1 This theorem gives a clue to the determination of A^{-1}, when it exists. $A^{-1} = \frac{1}{\det A}(\operatorname{adj} A)$

Note 2 The inverse of cA, where c is a non-zero real number and A is non-singular, is

$c^{-1}A^{-1}$ since $(cA)(c^{-1}A^{-1}) = (c^{-1}A^{-1})(cA) = I_n$

Note 3 Since $I_n \cdot I_n = I_n \cdot I_n = I_n$, it follows that $I_n^{-1} = I_n$. It follows that the inverse of the matrix cI_n is $c^{-1}I_n$ when $c \neq 0$.

Worked Example

1. If $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}$, find A^{-1}.

Solution: $\det A = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 2 \neq 0$. Since A is non-singular, A^{-1} exists and $A^{-1} = \frac{1}{\det A}(\operatorname{adj} A)$.

We have $\operatorname{adj} A = \begin{pmatrix} 1 & 3 & -4 \\ 2 & -2 & 2 \\ 1 & -3 & 4 \end{pmatrix}$

So, $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & -2 \\ 1 & 1 & -1 \\ \frac{1}{2} & -\frac{3}{2} & 2 \end{pmatrix}$

Theorem 5.3 If A and B be invertible matrices of the same order then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Scanned with CamScanner

Proof: Let A, B be $n \times n$ matrices.

Since A is invertible, A^{-1} exists and $AA^{-1} = A^{-1}A = I_n$ and also $\det A \neq 0$. Similarly, B^{-1} exists and $BB^{-1} = B^{-1}B = I_n$ and also $\det B \neq 0$.

We have $\det(AB) = \det A \cdot \det B \neq 0$

This shows that AB is non-singular and hence it is invertible.

Now $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = (AI_n)A^{-1} = AA^{-1} = I_n$

and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = (B^{-1}I_n)B = B^{-1}B = I_n$

So, we have $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n$.

From the definition of an inverse and its uniqueness, it follows that $B^{-1}A^{-1}$ is the inverse of AB. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

Corollary 1 If $A_1, A_2, \ldots, A_p$ be invertible matrices of the same order, $(A_1A_2\cdots A_p)^{-1} = A_p^{-1}A_{p-1}^{-1}\cdots A_1^{-1}$

Corollary 2. If A be an invertible matrix and p be a positive integer, $[A.A.\cdots A \text{ (p factors)}]^{-1} = A^{-1}.A^{-1}.\cdots A^{-1}$ (p factors)

That is, $(A^p)^{-1} = (A^{-1})^p$.

Theorem 5.4. If A be an invertible matrix then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

Scanned with CamScanner