So every minor of order $r+2$ is also zero because each such can be expressed ~~as~~ in terms of minors of order $(r+1)$. By similar arguments it can be proved that every of order greater than r is zero.

For a non-zero matrix A of order $m\times n$,
$$0 < \text{rank of } A \leq \min\{m,n\} \quad \left[\min\{m,n\} = \text{minimum of } m, n\right]$$

For a square matrix of order n, rank of $A < n$ or $= n$ according as A is singular or non-singular.

Rank of A = Rank of A^t, because A and A^t have identical ~~p~~ minors.

Examples 1. Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{pmatrix}$

Since $\det A = 0$, rank of $A < 3$. Now the second order minor $\begin{vmatrix} 1 & 0 \\ 4 & -1 \end{vmatrix} = -1 \neq 0$, rank of $A = 2$

2. Let $A = \begin{pmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{pmatrix}$

Solution: Here every minor of order 3 is zero here, because the third row is a multiple of the first and therefore rank of $A < 3$. There is a second order minor $\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} \neq 0$ and so rank of $A = 2$.

The rank of a matrix depends on the nature of its minors. The process of determining the rank of a matrix ~~...~~ starts from the evaluation of the minor of or minors of highest possible order and continues until a non-zero minor is obtained. So, the ~~...~~ method of determining a matrix of large order is much laborious and therefore we need some easier method for this purpose. With this aim in view, we now introduce some transformations on the matrix, called elementary operations on the matrix.

6.1 Elementary Operations

An elementary operation on a matrix A is an operation of the following types.

1. Interchange of two rows (or columns) of A
2. Multiplication of a row (or column) by a non-zero real number
3. ~~Add~~ Addition of a ~~real~~ multiple of one ~~row by real number~~ row (or column) by a real number

to another row (or column)

When applied to rows, the elementary operations are said to be elementary row operations and when applied to columns they are said to be elementary column operation.

The interchange of ith and jth row is denoted by R_{ij}

Multiplication of the ith row by a non-zero real number c is denoted by cR_i or $R_i(c)$

Addition of c times the jth row to the ith row is denoted by $R_i + cR_j$ or $R_{ij}(c)$

In a similar manner, the corresponding elementary column operations are denoted by C_{ij}, cC_i or $C_i(c)$, $C_i + cC_j$ or $C_{ij}(c)$

If T be an elementary operation on the matrix A, the transformed matrix is denoted by T(A) If $B = T(A)$ the operation is expressed as $A \xrightarrow{T} B$.

For examples

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 7 \\ 4 & 9 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & 4 & 0 \\ 4 & 9 & 5 \\ 1 & 3 & 7 \end{pmatrix} \xrightarrow{C_{23}} \begin{pmatrix} 2 & 0 & 4 \\ 4 & 5 & 9 \\ 1 & 7 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{2R_3} \begin{pmatrix} 2 & 1 & 3 \\ 4 & 5 & 0 \\ 6 & 4 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{2C_3} \begin{pmatrix} 2 & 1 & 6 \\ 4 & 5 & 0 \\ 3 & 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 4 & 9 \\ 1 & 3 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 4 \\ 4 & 9 \\ 1 & 3 \end{pmatrix} \xrightarrow{C_2 - 2C_1} \begin{pmatrix} 2 & 0 \\ 4 & 1 \\ 1 & 1 \end{pmatrix}$$

If T be an elementary ~~row (column)~~ operation on A such that $T(A) = B$ and T_1 be an elementary operation on B such that $T_1(B) = C$ then $C = T_1(T(A))$. So, C is obtained from A by applying two elementary operations T and T_1 successively.

If T be an elementary ~~operation~~ row (or column) operation on a matrix A then T^{-1}, the inverse of T, is defined to be an elementary row (column) operation such that $T^{-1}(TA) = A$

For example, if $T = R_{ij}$, $T^{-1} = R_{ij}$,

if $T = R_i(c)$ then $T^{-1} = R_i(c^{-1})$ (as $c \neq 0$)

if $T = R_{ij}(c)$ then $T^{-1} = R_{ij}(-c)$

Similarly for the column operations, T^{-1} can be defined.

~~Clearly~~ Clearly, the inverse of an elementary row (column) operation is an elementary row (column) operation of the same type.

6.2 Row equivalence. Column Equivalence

Let us consider the set S of all $m \times n$ real