matrices. A matrix B in S is said to be row equivalent (column equivalent) to a matrix A in S if B can be obtained by successive application of a finite number of elementary row operations (column operations) on A.

The relation of row equivalence (column equivalence) on the set S is an equivalence relation. Consequently, the set S is partitioned into classes of row equivalent (column equivalent) matrices.

We shall now discuss some properties of row equivalent matrix. Analogous properties hold in case of column equivalent matrices.

Definition : An $m \times n$ matrix A is called row-reduced if

(a) the first non-zero element in each non-zero row is 1 (called the leading 1), and

(b) in each column containing the leading 1 of some row, the leading 1 is the only non-zero element.

Examples of a row-reduced matrix are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Definition</u> An $m \times n$ matrix A is said to be a row-reduced echelon form (or a row echelon form) if

(a) A is row-reduced

(b) there is an integer r $(0 \le r \le m)$ such that the first r rows of A are non-zero rows and the remaining rows (if there be any) are all zero rows, and

(c) if the leading element of the ith non-zero row occurs in the k_ith column ~~a~~ of A, then

$k_1 < k_2 < \cdots < k_r$.

Examples of a row-reduced echelon matrix are

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We state some ~~a~~ theorems without proof

Theorem 6.2 An $m \times n$ matrix A can be made row equivalent to row-reduced echelon form by elementary row operations

Theorem 6.3 If a row-reduced echelon matrix R has r non-zero rows then rank of $R = r$

Theorem 6.4 The rank of a matrix remains invariant

Theorem 6.5 ~~An n×n matrix A is ~~ ~~Singular~~

If a matrix A ~~A~~ be row equivalent to a ~~row-reduced~~ row echelon form having r non-zero rows, then

rank of $A = r$

Examples: 1. ~~Obtain~~ Apply elementary operations to reduce the following matrix to a row echelon matrix and find its ~~r~~ rank :

$$\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}.$$

Solution

$$\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_3-5R_1]{R_2-3R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_4-3R_2]{R_3-2R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_{34}} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R \text{ (say)}.$$

R is a row echelon matrix and number of non-zero rows of R is 3. So rank of $R = 3$ and hence rank of $A = 3$.

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2. Obtain a row echelon matrix which is row equivalent to $\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$ and find the rank of the given matrix.

Solution: Let us apply ~~row~~ elementary row operation on the given matrix.

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

$$\xrightarrow[R_4-3R_1]{R_3-2R_1} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow[\substack{R_3+2R_2 \\ R_4+5R_2}]{R_1-2R_2} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \end{pmatrix} \xrightarrow{R_{34}} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2-R_3]{R_1-2R_3} \begin{pmatrix} 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$= R$ (say). R is row equivalent to the given matrix.