and it is a row echelon matrix. As rank of R is 3 (as it has 3 non-zero rows), so, the rank of the given matrix is 3.

6.3 System of linear equations:

A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \cdots (1)$$

where a_{ij}'s and b_i's are real numbers

If $x_1 = c_1, x_2 = c_2, \ldots, x_n = c_n$ satisfy each of the equations of (1), it is said to be a solution of the system (1) and we write $(c_1, c_2, c_3, \ldots, c_n)$ is a solution of the system (1)

Let $A = (a_{ij})_{m \times n}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

Then the system (1) can be written in matrix form as

$$AX = B \quad \text{---} (2)$$

If we write $A_i = (a_{i1}, a_{i2}, \ldots a_{in})$

then the ith equation of (1) can be written as

$$A_i X = b_i \quad , \quad i = 1, 2, \ldots m$$

Each equation $A_i X = b_i$ is called a vector equation and these m vector equations forms the system of linear equations (1) in the matrix form $AX = B$. where A_i are the rows of A, $i = 1, 2, \ldots, m$

In the system of linear equation $AX = B$, the matrix A is called the coefficient matrix of the system and the matrix, denoted by $[A, b]$ (or $\bar{A}$), and defined by
$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$
is said to be augmented matrix of the system.

The system $AX = B$ is said to be a homogeneous system if $B = 0$, otherwise it is said to be a non-homogeneous equation.

A system is said to be consistent if it has a solution. Otherwise, it is said to be inconsistent

Examples

1. $x_1 + 2x_2 = 3$
   $3x_1 + x_2 = 4$

(1, 1) is a solution of the system. There is no other solution of the system. This is a consistent system.

2. $x_1 + 2x_2 = 3$
   $3x_1 + 6x_2 = 7$

The system has no solution. This is not a consistent system.

$x_1 + 2x_2 - x_3 = 0$

$2x_1 + x_2 - 2x_3 = 0$

$(1,0,1)$ is a solution of the system, $(2,0,2)$, $(3,0,3)$ are also solutions. In fact, ~~(k,0,k)~~ $(k,0,k)$ is a solution for every real number k. Thus the system has many solutions.

Two systems $AX = B$ and $CX = D$ are said to be equivalent systems if the augmented matrices $[A, B]$ and $[C, D]$ are row equivalent.

We state a Theorem without proof.

Theorem 6.3.1 Let $AX = B$ and $CX = D$ be two equivalent systems and $(c_1, c_2, \ldots, c_n)$ is a solution of $AX = B$. Then $(c_1, c_2, \ldots, c_n)$ is also a solution of $CX = D$.

Corollary: If one of the two equivalent system be inconsistent then other is also so.

To examine the solvability of the system of linear equations $AX = B$ or to determine the solution (or solutions) of the system, when it is consistent, the best procedure is to apply such elementary row operations on the augmented matrix $[A, b]$ that will reduce it to a row-reduced echelon matrix.

Worked Examples

1. Solve the system of equations

$$x_1 + x_2 = 4$$
$$x_2 - x_3 = 1$$
$$2x_1 + x_2 + 4x_3 = 7$$

Solution: This is a non-homogeneous system. The coefficient matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 4 \end{pmatrix}$ and the augmented matrix $[A, B] = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 4 & 7 \end{pmatrix}$

where the system is $AX = B$.

Let us apply elementary row operations on $[A, B]$

$$[A, B] \xrightarrow{R_3 - 2R_1} \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 4 & -1 \end{pmatrix} \xrightarrow[R_3 + R_2]{R_1 - R_2} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow[R_2 + R_3]{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Hence the system is equivalent to
$$x_1 = 3$$
$$x_2 = 1$$
$$x_3 = 0.$$

and therefore the solution is $(3, 1, 0)$.