Note: The coefficient matrix A is row equivalent to the identity matrix $I_3$ and so it is non-singular. This also suggests that the system admits of a unique solution.

Solve the system of equations

$$x_1 + 3x_2 + x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0$$

Solution: The coefficient matrix of the system is

$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix}$ and the Augmented matrix

$$[A, B] = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 2 & -1 & 1 & 0 \end{pmatrix}$$

Let us apply elementary row operation on $[A, B]$

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ 2 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -7 & -1 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{7}R_2} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{7} & 0 \end{pmatrix}$$

~~The given syst~~ $\xrightarrow{R_1 - 3R_2} \begin{pmatrix} 1 & 0 & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & 0 \end{pmatrix}$

The given system is equivalent to

$$x_1 + \frac{4}{7}x_3 = 0$$
$$x_2 + \frac{1}{7}x_3 = 0$$

Assigning to $x_3$ an arbitrary real number $c$, we have the solution is $x_1 = -\frac{4}{7}c$, $x_2 = -\frac{1}{7}c$, $x_3 = c$, for every real number $c$. So, the system has infinite number of solutions.

Note: Instead of transforming the augmented matrix we can consider only the coefficient matrix A in case of a homogeneous system $Ax = 0$, since the last column of the augmented matrix is the zero column and this column remains unchanged under elementary row operations.

3. Solve, if possible, the system of equations

$$x_1 + 2x_2 - x_3 = 10$$
$$-x_1 + x_2 + 2x_3 = 2$$
$$2x_1 + x_2 - 3x_3 = 2$$

Solution: The coefficient matrix is $A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 1 & -3 \end{pmatrix}$

and the augmented matrix is $[A, B] = \begin{pmatrix} 1 & 2 & -1 & 10 \\ -1 & 1 & 2 & 2 \\ 2 & 1 & -3 & 2 \end{pmatrix}$

Let us apply elementary row operations on $[A, B]$

$$\begin{pmatrix} 1 & 2 & -1 & 10 \\ -1 & 1 & 2 & 2 \\ 2 & 1 & -3 & 2 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + R_1} \begin{pmatrix} 1 & 2 & -1 & 10 \\ 0 & 3 & 1 & 12 \\ 0 & -3 & -1 & -18 \end{pmatrix}$$

$$\xrightarrow{R_2 + R_3} \begin{pmatrix} 1 & 2 & -1 & 10 \\ 0 & 0 & 0 & -6 \\ 0 & -3 & -1 & -18 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 1 & 2 & -1 & 10 \\ 0 & -3 & -1 & -18 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

$$\xrightarrow[-\frac{1}{6}R_3]{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & -1 & 10 \\ 0 & 1 & 1/3 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -5/3 & -2 \\ 0 & 1 & 1/3 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 - 6R_3]{R_1 + 2R_3} \begin{pmatrix} 1 & 0 & -5/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The given system is equivalent to

$$x_1 - \tfrac{5}{3} x_3 = 0$$

$$x_2 + \tfrac{1}{3} x_3 = 0$$

and $\quad 0 = 1$

The last equation disallows the existence of any solution of the equivalent system.

Therefore, the given system is inconsistent

4. Solve, if possible, the system of linear equations

$$x_1 + 2x_2 - x_3 = 10$$

$$-x_1 + x_2 + 2x_3 = 2$$

$$2x_1 + x_2 - 3x_3 = 8$$

Solution: Let us apply elementary row operations on the augmented matrix of the system.

$$\begin{pmatrix} 1 & 2 & -1 & 10 \\ -1 & 1 & 2 & 2 \\ 2 & 1 & -3 & 8 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + R_1} \begin{pmatrix} 1 & 2 & -1 & 10 \\ 0 & 3 & 1 & 12 \\ 0 & -3 & -1 & -12 \end{pmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 2 & -1 & 10 \\ 0 & 3 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{pmatrix} 1 & 2 & -1 & 10 \\ 0 & 1 & 1/3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -5/3 & 2 \\ 0 & 1 & 1/3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The given system is equivalent to

$$x_1 - \frac{5}{3}x_3 = 2$$

$$x_2 + \frac{1}{3}x_3 = 4$$

Assigning ~~as~~ to $x_3$, an arbitrary ~~value of~~ real number $c$, the solution is

$$x_1 = 2 + \frac{5}{3}c$$

$$x_2 = 4 - \frac{1}{3}c$$

$$x_3 = c$$

When $c = 0$, we get a particular solution $x_1 = 2$, $x_2 = 4$ and $x_3 = 0$ of the given system.

and $x_1 = \frac{5}{3}c$, $x_2 = -\frac{1}{3}c$, $x_3 = c$, is the general solution of the homogeneous system corresponding to the given system.

Now, without going into the details, we give some related informations.

Existence and number of solution of a non-homogeneous system of linear equations $AX = B$, where $A$ is an $m \times n$ matrix.

Case I $m = n$

The system is consistent if and only if rank of $A$ = rank of $[A, B]$

For a consistent system, two cases arise.