Subcase (i) Rank of A = Rank of $[A, B] = n$

Here A is non-singular and so A^{-1} exists.

The system possesses the unique solution $X = A^{-1}B$

Subcase (ii) Rank of A = Rank of $[A, B] < n$

The associated homogeneous system $AX = 0$ has infinitely many solution and therefore the system $AX = B$ possesses infinitely many solution

(Note: when $m = n$, if rank of $A = n$, then $AX = 0$ possesses the unique solution $X = 0$ and if rank of $A < n$ then $AX = 0$ has infinitely many solutions)

Case 2: $m < n$

The system is consistent if and only if rank of A = rank of $[A, B] \le m$. If consistent rank of A = rank of $[A, B] < n$

In the consistent case, the homogeneous system $AX = 0$ has infinitely many solutions and therefore the system $AX = B$ possesses infinitely many solutions

(Note when $m < n$ and rank of $A < n$ then $AX = 0$ has infinitely many solutions)

Case 3 $m > n$

The system is consistent if and only if

rank of A = rank of $[A, B] \le n$

For a consistent system, two cases arise.

Suppose (i) rank of A = rank of $[A, B] = n$

The ~~system~~ homogeneous system $AX = 0$ possesses only the zero solution and therefore the system $AX = B$ possesses only one solution

(Note: when $m > n$ and rank of A $= n$ then $AX = 0$ has only zero solution)

Subcase (ii) rank of A = rank of $A < n$.

The associated homogeneous $AX = 0$ possesses infinitely many solutions and therefore the system $AX = B$ possesses infinitely many solution

(Note: when $m > n$ and rank of $A < n$ then $AX = 0$ has infinitely many solutions).

(Note: General solution of $AX = B$ is the General solution of $AX = 0$ + a particular solution of $AX = B$)

Worked Examples (continued) :

5. Solve, if possible

(i)
$$x+2y+z-3w=1$$
$$2x+4y+3z+w=3$$
$$3x+6y+4z-2w=5$$

(ii)
$$x+2y+z-3w=1$$
$$2x+4y+3z+w=3$$
$$3x+6y+4z-2w=4$$

Solution:

(i) This is a non-homogeneous system.

The coefficient matrix of the system $A = \begin{pmatrix} 1 & 2 & 1 & -3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 4 & -2 \end{pmatrix}$ and

the augmented matrix is $[A,B] = \begin{pmatrix} 1 & 2 & 1 & -3 & 1 \\ 2 & 4 & 3 & 1 & 3 \\ 3 & 6 & 4 & -2 & 5 \end{pmatrix}$

Let us apply elementary row operations on $[A,B]$

$$[A,B] \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & 1 & -3 & 1 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7 & 2 \end{pmatrix} \xrightarrow[R_3-R_2]{R_1-R_2} \begin{pmatrix} 1 & 2 & 0 & -10 & 0 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2-R_3} \begin{pmatrix} 1 & 2 & 0 & -10 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The rank of $[A,B]=3$ and rank of $A=2$

So, the system is inconsistent.

(ii) This is a non-homogeneous system. The coefficient matrix of the system $A = \begin{pmatrix} 1 & 2 & 1 & -3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 4 & -2 \end{pmatrix}$

and the augmented matrix $[A, B] = \begin{pmatrix} 1 & 2 & 1 & -3 & 1 \\ 2 & 4 & 3 & 1 & 3 \\ 3 & 6 & 4 & -2 & 4 \end{pmatrix}$

Let us apply elementary row operations on $[A, B]$

$$[A,B] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & -3 & 1 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7 & 1 \end{pmatrix} \xrightarrow[R_3 - R_2]{R_1 - R_2} \begin{pmatrix} 1 & 2 & 0 & -10 & 0 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here ~~rank [A, B]~~ rank of $[A, B]$ = rank of $A = 2$,

So, the system is consistent.

The given system is equivalent to

$$x + 2y - 10w = 0$$
$$z + 7w = 1$$

Choosing $y = c$, $w = d$, where c, d are arbitrary real numbers, the solutions are

$$x = -2c + 10d$$
$$y = c$$
$$z = 1 - 7d$$
$$w = d$$

, c, d are arbitrary real numbers.

7. Solve the system of equations in integers

$$x + 2y + z = 1$$
$$3x + y + 2z = 3$$
$$x + 7y + 2z = 1$$