2. Let X be a non-empty set. The relation ρ defined on $P(X)$ by $A \rho B$ if and only if A is a subset of B for $A, B \in P(X)$ ($P(X)$ is the power set of X, that is, the set of all subsets of X) is antisymmetric.

3. The relation R defined on $\mathbb{Z}$ by aRb if and only if a is a divisor of b. Then R is not antisymmetric as $1R(-1)$ and $(-1)R1$ but $1 \neq -1$.

1.3.1. Definition: Let S be a non-empty set. A relation ρ on S is said to be a partial order relation if ρ is reflexive, antisymmetric and transitive.

A relation of partial order is often denoted by '$\leq$' in any set.

1.3.2 Definition (Poset): A non-empty set S together with a relation of partial order $\leq$ on S is called a Poset (Partially ordered set) and is denoted by $(S, \leq)$

Examples (continued):

4. $(\mathbb{R}, \leq)$ is a poset where $x \leq y$ means x is less than or equal to y for $x, y \in \mathbb{R}$

5. Let X be a non-empty set and $P(X)$ be the power set of X. $(P(X), \leq)$ is a poset where $A \leq B$ means A is a subset of B for $A, B \in P(X)$

6. $(\mathbb{N}, \leq)$ is a poset where $m \leq n$ means m is a divisor of n for $m, n \in \mathbb{N}$

7. Let S be a positive divisor of of 72. $(S, \leq)$ is a poset where $a \leq b$ means a is a divisor of b for $a, b \in S$

Worked Example: 1. Let $(S, \leq)$ be a poset. Define a relation $\geq$ on S by $a \geq b$ if and only if $b \leq a$. Show that $(S, \geq)$ is a poset.

Solution: Since $a \leq a$ $\forall a \in S$. So, $a \geq a$ $\forall a \in S$. So, $\geq$ is reflexive. Let $a, b \in S$ and $a \geq b$ and $b \geq a$. Then $b \leq a$ and $a \leq b$. So, $b = a$ as $\leq$ is antisymmetric.

So, $a \geq b$ and $b \geq a \Rightarrow a = b$. So, $\geq$ is antisymmetric.

Let $a, b, c \in S$ and $a \geq b$ and $b \geq c$. Then $b \leq a$ and $c \leq b$. So, $c \leq b$ and $b \leq a$. This implies $c \leq a$ as $\leq$ is transitive. So, $a \geq c$. So, $a \geq b$ and $b \geq c \Rightarrow a \geq c$. Hence $\geq$ is transitive.

Hence $\geq$ is a partial order relation on S and $(S, \geq)$ is a poset.

Note 1. The poset $(S, \geq)$ is called the dual of the poset $(S, \leq)$

Note 2. In a definition, 'if and only if' and 'if' are used in the same sense. So, henceforth, we will only use 'if'.

2. Let $\rho = \{(a, b) \in \mathbb{R} \times \mathbb{R} : a - b \leq 0\}$ be a relation on $\mathbb{R}$. Show that ρ is a partial order relation on $\mathbb{R}$.

Solution: As $a - a = 0 \leq 0$ $\forall a \in \mathbb{R}$, so $a \rho a$ ~~is reflexive~~ $\forall a \in \mathbb{R}$

Hence ρ is reflexive.

Let $a, b \in \mathbb{R}$ and $a \rho b$ and $b \rho a$. So, $a - b \leq 0$ and $b - a \leq 0$. So, $a \leq b$ and $b \leq a$. Hence, $a = b$ as $\leq$ is partial order relation on $\mathbb{R}$.

Hence $a \rho b$ and $b \rho a \Rightarrow a = b$. Hence, ρ is antisymmetric.

Let $a, b, c \in \mathbb{R}$ and $a\rho b$ and $b\rho c$. This implies

$a-b \leq 0$ and $b-c \leq 0$. So, $a \leq b$ and $b \leq c$

So, $a \leq c$ as $\leq$ is transitive. So, $a-c \leq 0$

So, $a\rho c$. Hence, ρ is transitive.

Hence, ρ is a partial order relation on $\mathbb{R}$.

1.3.2 Definition: A partial order relation ρ on a non-empty set S is said to be a linear order if $\forall a, b \in S$, either $a\rho b$ or $b\rho a$. and (S, ρ) is said to be a linearly ordered set or a chain.

Examples: In Examples 4 of Page 9, $(\mathbb{R}, \leq)$ is a linearly ordered set. but in Example 5, $(P(X), \leq)$ is not a linearly ordered set. For Example, if $X = \{1, 2, 3, 4\}$, then $\{1\} \in P(X)$ and $\{2, 3\} \in P(X)$ but $\{1\}$ is not a subset of $\{2, 3\}$ and $\{2, 3\}$ is not a subset of $\{1\}$.

We end our discussion on relation by merely mentioning that the idea of a Poset paves the way towards a very important branch of mathematic, known as lattice theory.

Worked out exercises 1.4:

1.4.1 For each of the following relations ρ on the set $\mathbb{Z}$, examine whether it is reflexive, symmetric, antisymmetric or transitive:

(i) $a\rho b$ if $a \leq b$ for $a, b \in \mathbb{Z}$

(ii) $a\rho b$ if $ab \geq 0$ for $a, b \in \mathbb{Z}$

(iii) $\rho = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : a^2+b^2$ is a multiple of $2\}$

(iv) $a\rho b$ if $a \in \mathbb{Z}, b \in \mathbb{Z}$ and $2a+3b$ is divisible by 5

Solution: (i) As $a \leq a$ for all $a \in \mathbb{Z}$, $a\rho a$ $\forall a \in \mathbb{Z}$. So, ρ is reflexive.

~~Let $a\rho b$ and $b\rho a$~~ Now, $2 \rho 3$ as $2 \leq 3$ but $3 \not\rho 2$ as $3 \not\leq 2$. So, ρ is not symmetric.

Let $a\rho b$ and $b\rho a$ for $a, b \in \mathbb{Z}$. So, $a \leq b$ and $b \leq a$. Hence $a = b$. So, ρ is antisymmetric.

Let $a\rho b$ and $b\rho c$, for $a, b, c \in \mathbb{Z}$. Then $a \leq b$ and $b \leq c$. So, $a \leq c$. Hence $a\rho c$. So, ρ is transitive.

(ii) $a\rho a$ $\forall a \in \mathbb{Z}$ as $a^2 \geq 0$ $\forall a \in \mathbb{Z}$. So, ρ is reflexive.

Let $a\rho b$, for $a, b \in \mathbb{Z}$. Then $ab \geq 0$. So, $ab = ba \geq 0$. Hence, $b\rho a$. So, ρ is symmetric.

(iii) and (iv) are homeworks.

1.4.2. For each of the following relations ρ on $A = \{1, 2, 3, 4\}$, examine whether it is reflexive, symmetric, antisymmetric or transitive.

(i) $\rho = \{(1,3), (3,1)\}$

(ii) $\rho = \{(2,2)\}$

(iii) $\rho = \{(1,2), (1,4)\}$

(iv) $\rho = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2)\}$