Solution: This is a non-homogeneous system of equations.

The augmented matrix is $[A, B] = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & 3 \\ 1 & 7 & 2 & 1 \end{pmatrix}$

Let us apply the elementary row operations on $[A, B]$

$$[A,B] \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & 5 & 1 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{5}R_2} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 5 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[R_3 - 5R_2]{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 3/5 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The given system is equivalent to

$$x + \tfrac{3}{5} z = 1$$

$$y + \tfrac{1}{5} z = 0$$

Choosing $z = c$, the solution is

$$x = 1 - \frac{3}{5}c, \quad y = 0 - \frac{1}{5}c, \quad z = c, \quad c \in \mathbb{R}.$$

Since the solutions are to be integers, $c = 5k$ where $k$ is an arbitrary integer.

Hence the solution is

$x = 1 - 3k$, $y = -k$ and $z = 5k$, $k$ being an integer.

8. Determine the condition for which the system

$x+y+z=1$

$x+2y-z=b$

$5x+7y+az=b^2$

admits of (i) only one solution, (ii) no solution, (iii) many solutions.

Solution: The system has a unique solution if the coefficient determinant be non-zero

The Coefficient determinant $= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & a \end{vmatrix} = a-1$.

If $a-1 \neq 0$, i.e., $a \neq 1$, the system has only one solution

If $a=1$, the system has either no solution or many solution, When $a=1$, the coefficient matrix of the system is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & 1 \end{pmatrix}$

and the augmented matrix of the system is $[A,B] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & 1 & b^2 \end{pmatrix}$

Let us apply elementary row operations on $[A,B]$

$$[A,B] \xrightarrow[R_3-5R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 2 & -4 & b^2-5 \end{pmatrix} \xrightarrow[R_3-2R_2]{R_1-R_2} \begin{pmatrix} 1 & 0 & 3 & -b+2 \\ 0 & 1 & -2 & b-1 \\ 0 & 0 & 0 & b^2-2b-3 \end{pmatrix}$$

If $b^2-2b-3=0$, then rank of $[A,B]$ = rank of $A = 2$ and therefore the system is consistent and as rank of A = rank $[A,B] = 2 < 3$ = number of variable, the system has many solution.

So, if $a=1$ and $b^2-2b-3=0$, the system has many solution.

i.e., if $a=1$, $b=-1$ or if $a=1$, $b=3$, the system has many solution as $b^2-2b-3=(b+1)(b-3)$

If $b^2-2b-3 \neq 0$, then rank of $[A;B] = 3$ and rank of $A = 2$ and since rank $A \neq$ rank of $[A,B]$, the system is inconsistent, i.e it has no solution.

So, If $a=1$ and $b^2-2b-3 \neq 0$, the system has no solution.

7. Examine whether the following system is consistent or not and if consistent, find ~~the~~ all the solutions of the system:

$$x+2y-z=6$$
$$3x-y-2z=3$$
$$4x+3y+z=9$$

Here the augmented matrix is $[A, B] = \begin{pmatrix} 1 & 2 & -1 & 6 \\ 3 & -1 & -2 & 3 \\ 4 & 3 & 1 & 9 \end{pmatrix}$

Applying elementary row operations on $[A, B]$, we have

$$[A,B] \xrightarrow[R_3-4R_1]{R_2-3R_1} \begin{pmatrix} 1 & 2 & -1 & 6 \\ 0 & -7 & 1 & -15 \\ 0 & -5 & 5 & -15 \end{pmatrix}$$

$$\xrightarrow{R_2-R_3} \begin{pmatrix} 1 & 2 & -1 & 6 \\ 0 & -2 & -4 & 0 \\ 0 & -5 & 5 & -15 \end{pmatrix} \xrightarrow[-\frac{1}{5}R_3]{-\frac{1}{2}R_2} \begin{pmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow[R_3-R_2]{R_1-2R_2} \begin{pmatrix} 1 & 0 & -5 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \xrightarrow{-\frac{1}{3}R_3} \begin{pmatrix} 1 & 0 & -5 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow[R_2-2R_3]{R_1+5R_3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

So, the rank of the coefficient matrix

$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{pmatrix}$ is 3

and rank of $[A, B] = 3$. So rank of $A$ = rank of $[A, B] = 3$

So, the system is consistent

The system is equivalent to