Operator.

The following properties of the operator Δ can be easily established from the definition:

(i) $\Delta c = 0$, c is a constant

(ii) $\Delta\{f(x) \pm g(x)\} = \Delta f(x) \pm \Delta g(x)$

(iii) $\Delta\{f(x).g(x)\} = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

(iv) $\Delta a^x = a^x(a-1)$, where $h=1$

(v) $\Delta^m\Delta^n f(x) = \Delta^n\Delta^m f(x) = \Delta^{m+n} f(x)$, where m, n are positive integers and $h=1$

[Note: We define $\Delta^k f(x) = \Delta(\Delta^{k-1} f(x))$, k is a positive integer
and define $\Delta^0 f(x) = I f(x) = f(x)$ (I is the identity operator, sometimes written as 1)]

Let us prove the result (iii) above.

$$\Delta\{f(x).g(x)\} = f(x+h)g(x+h) - f(x)g(x)$$
$$= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)$$
$$= f(x+h)\left(g(x+h) - g(x)\right) + g(x)\left(f(x+h) - f(x)\right)$$
$$= f(x+h)\Delta g(x) + g(x)\Delta f(x)$$

<u>Law of formation of successive differences</u>

$$\Delta f(x) = f(x+h) - f(x) = F(x) \text{ (say)}$$
$$\Delta^2 f(x) = \Delta\left(\Delta f(x)\right) = \Delta F(x) = \Delta f(x+h) - \Delta f(x)$$
$$= f(x+h+h) - f(x+h) - f(x+h) + f(x)$$
$$= f(x+2h) - {}^2c_1\, f(x+h) + f(x)$$

By induction, we can prove that for ~~an~~ any positive integer n,

$$\Delta^n f(x) = f(x+nh) - {}^nC_1 f(x+(n-1)h) + {}^nC_2 f(x+(n-2)h) - \cdots + (-1)^n f(x)$$

<u>Further discussion on differences</u>

Denoting the function of x by u_x, let $u_x = ax^k$, where a and k are constants, ~~with~~ and k is a positive integer.

Then $\Delta u_x = a\{(x+1)^k - x^k\}$ (taking $h=1$)

$$= a\left\{{}^kC_1 x^{k-1} + {}^kC_2 x^{k-2} + \cdots + 1\right\}$$

which is a polynomial in x of degree $k-1$.

Thus if u_x be a polynomial in x of degree k, Δu_x is a polynomial in x of degree $k-1$.

Hence it can be easily shown that $\Delta^2 u_x$ will be a polynomial in x of degree $k-2$, and so on.

Thus $\Delta^{k-1} u_x$ will be a polynomial of the form $bx+c$, where b and c are independent of x.

Hence $\Delta^k u_x$ will be a constant.

Now, if ~~a~~ $n > k$, $\Delta^n u_x = 0$

The converse of this proposition is also true as will be evident from the following example;

Consider a series $\sum u_n$, whose nth term is (n^3+5n^2+6n)

The series is then by putting $n=1,2,3,\ldots$ successively,

12, 40, 90, 168, 280, 432, ...

and the successive difference series are $(h=1)$

Δu_n 28 50 78 112 152 ...

$\Delta^2 u_n$ 22 28 34 40 ...

$\Delta^3 u_n$ 6 6 6 ...

$\Delta^4 u_n$ 0 0 ...

Thus $\Delta^3 u_n = 6$ and $\Delta^4 u_n = 0$ for all values of n

Conversely, if $\Delta^4 u_n = 0$ for all values of n, it can be shown that $\Delta^3 u_n$ is constant, $\Delta^2 u_n$ is of first degree, Δu_n is of second degree in n and u_n is cubic in n

Shift operator

From the relation $\Delta f(x) = f(x+h) - f(x)$, we have

$$f(x) + \Delta f(x) = f(x+h),$$

Which is $(1+\Delta) f(x) = f(x+h)$

In the operational form, writing E for $(1+\Delta)$, we have $E f(x) = f(x+h)$

Thus E may be regarded, as an operator, called shift operator, which operates on the function $f(x)$

to give, the value of the function at the point $(x+h)$, h being the usual positive increment in the independent variable x

We define, for any positive integer k

$$E^k f(x) = E\left(E^{k-1} f(x)\right) \quad \text{and} \quad E^0 f(x) = f(x)$$

So, for any positive integer n, we can show that

$$E^n f(x) = f(x+nh)$$

~~Definit~~ Defining the inverse operator E^{-1} by

$$E^{-1} f(x) = f(x-h)$$

we see that $\nabla \equiv 1 - E^{-1}$, while $\Delta \equiv E - 1$

The following properties of the shift operator can easily be established from definition :

(i) $E\{f(x) \pm g(x)\} = Ef(x) \pm Eg(x)$

(ii) $E\{cf(x)\} = cE\{f(x)\}$, c is a constant.

(iii) $E^m E^n f(x) = E^{m+n} f(x) = E^n E^m f(x)$

(iv) $\Delta E f(x) = E \Delta f(x)$

We know that $\Delta f(x) = f(x+h) - f(x)$

That is, $(1+\Delta) f(x) = f(x+h) = f(x) + \Delta f(x)$

$$(1+\Delta)^2 f(x) = (1+\Delta)(1+\Delta) f(x) = (1+\Delta)\left(f(x) + \Delta f(x)\right)$$

$$= f(x) + \Delta f(x) + \Delta f(x) + \Delta^2 f(x)$$

$$= f(x) + 2\Delta f(x) + \Delta^2 f(x) = \left(1 + 2\Delta + \Delta^2\right) f(x)$$