Difference equation may also be defined as an equation which expresses a relation between the independent variable and successive values of the dependent variable.

For example, the equation (assuming $h=1$)

$\Delta^3 u_x + 5\Delta^2 u_x + 4\Delta u_x + 7u_x = x^3 + 2x^2$ can be put as

$$u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x + 5u_{x+2} - 10u_{x+1} + 5u_x + 4u_{x+1} - 4u_x + 7u_x = x^3 + 2x^2$$

or, $u_{x+3} + 2u_{x+2} - 3u_{x+1} + 7u_x = x^3 + 2x^2 \quad \ldots (2)$

which involves the successive values of the dependent variable u_x at $x, x+1, x+2$ and $x+3$

Introducing the shift operator E, the equation (2) can be written as

$$E^3 u_x + 2E^2 u_x - 3E u_x + 7u_x = x^3 + 2x^2$$

The difference between the largest and the smallest arguments ~~appear~~ appearing in the difference equation with unit interval is called its order.

Thus the difference equation (2) ~~is of~~ above is a third order difference equation. While the order of the equation (1) is two. The equation

$f(x+3) - 4f(x+1) = 2^x$ is, in fact, of the second order as $(x+3) - (x+1) = 2$

The order of the equation $\Delta U_{x+1} - U_x = 0$ is two, while that of $U_{x+2} - U_{x-1} = 0$ is three

If $f(x) = 0$ in (1), the equation is homogeneous; otherwise it is a non-homogeneous equation.

If a difference equation be such that the coefficients of the successive differences are constants and the differences of successive orders are of first degree, then the equation is a linear difference equation with constant coefficients.

The equation $(\Delta U_x)^2 + \Delta^2 U_x + U_x^2 = 2$ is not linear.

A solution of a difference equation is a relation between the independent variable and the dependent variable, satisfying the equation.

Now let us see how the difference equation is formulated.

For that let $U_x = c a^x$ ... (3)

in which c is an arbitrary constant

Then we have $U_{x+1} = c a^{x+1} = a c a^x$ ... (4)

Eliminating c from (3) and (4), we get the linear difference equation $U_{x+1} = a U_x$ of the first order whose solution is (3) and which contains an arbitrary constant.

Consider again, $U_x = A a^x + B b^x$ ... (5) in which

A and B are arbitrary constants.

Then we write $U_{x+1} = A a^{x+1} + B b^{x+1}$ ... (6)

and $U_{x+2} = A a^{x+2} + B b^{x+2}$ ... (7)

From (5) and (6), we get

$$U_{x+1} - aU_x = B(b-a) b^x$$

From (6) and (7), we then get

$$U_{x+2} - aU_{x+1} = B(b-a) b^{x+1}$$

Thus, $U_{x+2} - aU_{x+1} = B(b-a) b^x . a = a (U_{x+1} - aU_x)$

or, $U_{x+2} - (a+b) U_{x+1} + ab U_x = 0$ ... (8)

which is a linear difference equation of order two whose solution is (5) containing two arbitrary constants

Linear difference equations with constant coefficients

(i) First order homogeneous equation

Consider the difference equation

$$U_{x+1} - aU_x = 0, \quad a \neq 1$$

It is easily seen that $U_x = ca^x$ is a solution of this difference equation, where c is an arbitrary constant.

It should be noted that the solution contains one arbitrary constant.

(ii) Second ~~grade~~ order homogeneous equations

We consider the second order equation with constant coefficients in the form $U_x + a u_{x-1} + b U_{x-2}$, $x \geq 3$

where a, b are constants.

Putting $x = 3$, $U_3 = -aU_2 - bu_1$ ...(1)

Again, putting $x = 4$, we have $u_4 = -au_3 - bu_2$

$$= a(au_2 + bu_1) - bu_2 \quad \text{from (1)}$$

$$= ab\,u_1 + (a^2 - b)u_2$$

Thus, we see that U_3 and U_4 can be expressed in the form $\lambda u_1 + \mu u_2$ where λ and μ are independent.

of u_1 and u_2

By induction, one can see that u_r can be expressed as

$$u_r = Au_1 + BU_2.$$

where A and B are independent of U_1 and U_2

Arbitrary values can be assigned to u_1 and u_2

and hence the general solution of the equation of

the second order will contain two arbitrary

constants

It is a property of the linear equations and is evident by direct substitution that $U_r = f(r)$ and $u_r = g(r)$ be two solutions and if A and B be two numbers independent of r, then $u_r = Af(r) + Bg(r)$ is also a solution. This contains two arbitrary constants

A solution involving as many arbitrary constants as is the order of the equation is called the general solution.