Solution: (i) Since $(1,1) \notin \rho$, ρ is not reflexive. Further, we see that whenever $(a,b) \in \rho$, we also have $(b,a) \in \rho$. So, ρ is symmetric. Now, $(3,1) \in \rho$, $(1,3) \in \rho$ but $(3,3) \notin \rho$ and so ρ is not transitive.
Finally, $(1,3) \in \rho$, $(3,1) \in \rho$ but $1 \neq 3$. Hence ρ is not antisymmetric.

(ii), (iii) and (iv) are homework.

1.4.3 The following relations are defined on the $\mathbb{R}$ of real numbers. Find whether these relations are reflexive, symmetric or transitive.
(i) $a \rho b$ if $|a-b| > 0$ (ii) $a R b$ if $1+ab > 0$ (iii) $a R b$ if $|a| \leq b$

Solution: (i) ρ is not reflexive, since for any $a \in \mathbb{R}$, $a - a = 0$ and hence $|a-a| \ngtr 0$. So $a \not\rho a$. So, ρ is not reflexive. Again, as $|a-b| = |b-a|$ for $a, b \in \mathbb{R}$, we have if $|a-b| > 0$ then $|b-a| > 0$. So, ρ is symmetric as $a \rho b$ implies $b \rho a$ for $a, b \in \mathbb{R}$. Here ρ is not transitive. For, $0, 1 \in \mathbb{R}$ and $|1-0| = |0-1| = 1 > 0$ shows that $1 \rho 0$ and $0 \rho 1$ but $1 \not\rho 1$ as $|1-1| = 0 \ngtr 0$

(ii) Since, $\forall a \in \mathbb{R}$, $a^2 \geq 0$. So, we have $1 + a^2 > 0$ $\forall a \in \mathbb{R}$. Hence, $a \rho a$ $\forall a \in \mathbb{R}$. So, ρ is reflexive. Let $a, b \in \mathbb{R}$ and $a \rho b$. This implies $1+ab > 0$. As $ab = ba$, we have $1 + ba = 1 + ab > 0$. So, $b \rho a$. Hence ρ is symmetric.
Now, consider $3, -\frac{1}{9}, -6 \in \mathbb{R}$. Then $1 + 3(-\frac{1}{9}) = 1 - \frac{1}{3} = \frac{2}{3} > 0$ shows that $3 \rho (-\frac{1}{9})$ and $1 + (-\frac{1}{9})(-6) = 1 + \frac{2}{3} = \frac{5}{3} > 0$ shows that $(-\frac{1}{9}) \rho (-6)$. Now $1 + 3(-6) = 1 - 18 = -17 \ngtr 0$. So, $3 \not\rho (-6)$. Hence ρ is not transitive.

(iii) Let us consider $-2 \in \mathbb{R}$. Then $|-2| = 2 \nleq -2$. So, $(-2) \not\rho (-2)$.
Hence ρ is not reflexive. Now $(-2) \rho 5$ as $|-2| = 2 \leq 5$.
But $5 \not\rho (-2)$ as $|5| = 5 \nleq -2$. So, ρ is not symmetric.
Let $a, b, c \in \mathbb{R}$ such that $a \rho b$ and $b \rho c$. So, $|a| \leq b$ and $|b| \leq c$. Now since $b \geq |a| \geq 0$, we have $|b| = b$.
So, $|a| \leq b \leq c$. So, $|a| \leq c$. So, $a \rho c$. Hence ρ is transitive.

1.4.4 In each of the following cases, examine whether the relation ρ is an equivalence relation on $\mathbb{Z}$ (the set of all integers):

(i) $\rho = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : |a-b| \leq 3\}$

(ii) $\rho = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a-b \text{ is a multiple of } 6\}$

Solution: Homework

1.4.5 Justify the following statements or else give counter-examples to disprove:
(An example that disproves a statement is called a Counterexample)

(i) The intersection of two equivalence relations is again an equivalence relation.

(ii) The union of two equivalence relations is again an equivalence relation.

Solution: (i) We prove that the statement is true.
Let R and S be two equivalence relations on a non-empty set A.
Let $T = R \cap S$. So, T is a relation on A.

Let $a \in A$. Then $aRa \; \forall a \in A$ and $aSa \; \forall a \in A$

That is, $(a,a) \in R \; \forall a \in A$ and $(a,a) \in S \; \forall a \in A$

So, $(a,a) \in R \cap S = T \; \forall a \in A$. So, T is reflexive

Let $a, b \in A$ and aTb. So, $(a,b) \in T = R \cap S$

This implies $(a,b) \in R$ and $(a,b) \in S$. So, $(b,a) \in R$ and $(b,a) \in S$ as R and S are symmetric.

So, $(b,a) \in R \cap S = T$. So, T is symmetric

Let $a, b, c \in A$ and $(a,b) \in T$ and $(b,c) \in T$

So, $(a,b) \in R$ and $(a,b) \in S$. Also, $(b,c) \in R$ and $(b,c) \in S$

So, $(a,b) \in R$ and $(b,c) \in R$. This implies $(a,c) \in R$ as R is transitive. Similarly as $(a,b) \in S$ and $(b,c) \in S$, $(a,c) \in S$ as S is transitive. Hence $(a,c) \in R \cap S = T$

So, T is transitive. So, $T = R \cap S$ is an equivalence relation.

(ii) The statement is false. Let $A = \{1,2,3\}$. We consider two equivalence relations R and S on A given by

$R = \{(1,1),(2,2),(3,3),(2,3),(3,2)$ and $S = \{(1,1),(2,2),(3,3),(3,1),(1,3)\}$

Here $R \cup S = \{(1,1),(2,2),(3,3),(2,3),(3,2),(3,1),(1,3)\}$

Here $(2,3),(3,1) \in R \cup S$ but $(2,1) \notin R \cup S$

So, $R \cup S$ is not transitive. So $R \cup S$ is not ~~as~~ an equivalence relation.

1.4.6 Show that the relation ρ on $\mathbb{Z}$ defined by $x \rho y \Leftrightarrow x^2 - y^2$ is divisible by 5, is an equivalence relation on $\mathbb{Z}$, $x, y \in \mathbb{Z}$. What are its equivalence classes

Solution: Let $a \in \mathbb{Z}$. Then $a^2 - a^2 = 0$ is divisible by 5. So, $a \rho a$ $\forall a \in \mathbb{Z}$. So, ρ is reflexive. Let $a, b \in \mathbb{Z}$ and $a \rho b$ then $a^2 - b^2$ is divisible by 5. So, $a^2 - b^2 = 5k$, $k \in \mathbb{Z}$

So, $b^2 - a^2 = 5(-k)$, $-k \in \mathbb{Z}$. So, $b^2 - a^2$ is divisible by 5. So, $b \rho a$. So, ρ is symmetric

Let $a, b, c \in \mathbb{Z}$ and $a \rho b$ and $b \rho c$. Then $a^2 - b^2$ is divisible by 5 and $b^2 - c^2$ is divisible by 5. So,

$a^2 - b^2 = 5k_1$ and $b^2 - c^2 = 5k_2$, $k_1, k_2 \in \mathbb{Z}$. So,

$a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2) = 5k_1 + 5k_2 = 5(k_1 + k_2)$, $k_1 + k_2 \in \mathbb{Z}$

So, $a^2 - c^2$ is divisible by 5. So, $a \rho c$. So, ρ is transitive. Hence, ρ is an equivalence relation on $\mathbb{Z}$

Here $cl(0) = \{x \in \mathbb{Z} : x \rho 0\} = \{x \in \mathbb{Z} : x^2 \text{ is divisible by } 5\}$

~~now~~ we know that ~~for~~ if a prime p divides ab then ~~p/a~~ p divides a or p divides b.

So, $x \in cl(0) \Rightarrow x^2$ is divisible by 5

$\Rightarrow$ 5 divides x. So, $x = 5k$, $k \in \mathbb{Z}$

So, $cl(0) = \{5k : k \in \mathbb{Z}\}$

If $x \in cl(1)$ then $x^2 - 1^2$ is divisible by 5

So, 5 divides $x-1$ or 5 divides $x+1$