So, $x = 5k+1$ or, $x = 5k-1 = 5k'+4$ for $k, k' \in \mathbb{Z}$
$k' = k-1$.

So, $cl(1) = \{5k+1 \text{ or } 5k+4 : k \in \mathbb{Z}\}$

If $x \in cl(2)$ then x^2-4 is divisible by 5.

So, 5 divides $x-2$ or 5 divides $x+2$.

So, $x = 5k+2$ or $x = 5k-2 = 5k'+3$ for, $k, k' \in \mathbb{Z}$
$k' = k-1$

So, $cl(2) = \{5k+2 \text{ or } 5k+3 : k \in \mathbb{Z}\}$

So, the equivalances classes are $cl(0)$, $cl(1)$ and $cl(2)$ as they cover all of $\mathbb{Z}$.

1.4.7 Examine if the relation ρ on the set $\mathbb{Z}$ is an equivalence relation.

(a) $\rho = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : 3a+4b \text{ is divisible by } 7\}$

(b) $\rho = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : |a-b| \leq 3\}$

Solution: (a) As $3a+4a = 7a$ is divisible by 7, $(a,a) \in \rho$ for all $a \in \mathbb{Z}$. So, ρ is reflexive.

Let $(a,b) \in \rho$. Then $3a+4b$ is divisible by 7, $a, b \in \mathbb{Z}$

So, $3a+4b = 7k$, $k \in \mathbb{Z}$

Now $3b+4a + 3a+4b = 7(a+b)$

$$
\begin{aligned}
\text{So, } \quad 3b+4a &= 7(a+b) - (3a+4b) \\
&= 7(a+b) - 7k \\
&= 7(a+b-k), \quad a+b-k \in \mathbb{Z}
\end{aligned}
$$

So, $3b+4a$ is divisible by 7. So, $(b,a) \in \rho$. Hence ρ is symmetric.

Let $(a,b) \in \rho$ and $(b,c) \in \rho$ for $a, b, c \in \mathbb{Z}$

So, $3a+4b = 7k_1$ and $3b+4c = 7k_2$ for, $k_1, k_2 \in \mathbb{Z}$

So, $3a+4b+3b+4c = 7(k_1+k_2)$

or, $3a+4c = 7(k_1+k_2-b)$, $k_1+k_2-b \in \mathbb{Z}$

So, $3a+4c$ is divisible by 7. So, $(a,c) \in \rho$.

So, ρ is transitive.

Hence ρ is an equivalence relation on $\mathbb{Z}$

(b) Here $(1,3) \in \rho$ as $|1-3| = 2 \leq 3$. Also $(3,5) \in \rho$ as $|3-5| = 2 \leq 3$. But $|1-5| = 4 \nleq 3$

So, $(1,5) \notin \rho$. So, ρ is not transitive.

Hence ρ is not an equivalence relation.

Exercises for Tutorial (1)

[Solutions to these exercises are to be submitted within January, 07, 2020 (07.01.2020)

Make a pdf file by scanning the solutions pages and send it by whatsapp. Strictly follow the time schedule for sending the solutions pages. If you need help for any particular problem, you can ask me by ~~whatt~~ whatsapp, I will give you some hints. But this ~~should~~ should be before ~~within~~ ~~before~~ 07.01.2020.]

T1. Find the number of reflexive relations on a set of 3 elements. Can you guess a formula for the number of reflexive relations on a set of n elements?

T2. For the partition $\mathcal{P} = \{\{a\}, \{b,c\}, \{d,e\}\}$, write the corresponding equivalence relation on the set $A = \{a, b, c, d, e\}$.

T3. Find out which of the following relation ρ are equivalence relations on the set $\mathbb{Z}$:

(i) $a \rho b$ if and only if $a^2 - b^2$ is a multiple of 7

(ii) $a \rho b$ if and only if $a^2 = b^2$

(iii) $a \rho b$ if and only if $a \leq |b|$

(iv) $a \rho b$ if and only if $a - b$ is an even integer

(v) $a \rho b$ if and only if $a - b$ is an odd integer

(vi) $a \rho b$ if and only if $|a| = |b|$

(vii) $a \rho b$ if and only if $b = a^2$

T4. Consider the following relations:

(i) the relation '$\perp$' (perpendicular) on the L of all straight lines in a plane

(ii) the relation '$\parallel$' (parallel) on the set L of all straight lines on a plane

(iii) relation $\rho_1 = \{(1,1), (2,2), (3,4), (4,3), (4,4)\}$ on $A = \{1,2,3,4\}$

(iv) relation '|' of divisibility among the set of natural numbers.

Find which of these relations are (a) reflexive (b) symmetric (c) transitive (d) antisymmetric.

T5. We define a relation ρ^{-1} on a non-empty set A by $\rho^{-1} = \{(a,b) : (b,a) \in \rho\}$ where ρ is a relation on A.
We also define a relation $\rho_1 \circ \rho_2$ on A, called the composition of ρ_1 and ρ_2, where ρ_1 and ρ_2 are relations on A, by

$$\rho_1 \circ \rho_2 = \{(a,c) \in A \times A : \text{there exists } b \in A \text{ such that } (a,b) \in \rho_1 \text{ and } (b,c) \in \rho_2\}.$$

Prove the following assertions if you can, or else give counterexamples to disprove:

(a) If ρ is a transitive relation on a set A, then so is ρ^{-1}.

(b) If ρ_1 and ρ_2 are transitive relations on a set A, then so is $\rho_1 \circ \rho_2$.

(c) Every relation must either be symmetric or antisymmetric.

(d) Let ρ be a reflexive and transitive relation on a set A. Then $\rho \cap \rho^{-1}$ is an equivalence relation.