2. Now we cover the portion of Unit 2 as mentioned in Page-1 of Notes.

The set $\{1, 2, 3, \ldots\}$ is said to be the set of all natural numbers or the set of all positive integers and is denoted by $\mathbb{N}$. The set $\{0, 1, -1, 2, -2, \ldots\}$ is said to be the set of all integers and is denoted by $\mathbb{Z}$

2.1 Well ordering property of $\mathbb{N}$

The well ordering property of $\mathbb{N}$ or positive integers states that every non-empty subset of $\mathbb{N}$ contains a least element.

This means that if S be a non-empty subset of $\mathbb{N}$ there is some natural number a in S such that $a \leq x$ for all $x \in S$.

2.2 Principles of Mathematical induction

There are two principles of Mathematical induction. One is called principle (or first principle) of Mathematical induction and the other is called Second principle of Mathematical induction.

2.2.1 Principle (or first principle) of Mathematical induction:

Let S be a subset of $\mathbb{N}$ with the properties:

(i) $1 \in S$

(ii) whenever a natural number $k \in S$, then $k+1 \in S$

Then $S = \mathbb{N}$.

Proof: Let T be the set of all those natural numbers which are not in S. The theorem will be proved if we can prove that T is an empty set.

Let us assume that T is a non-empty set. Then by the

well ordering property of $\mathbb{N}$, T has a least element, say m.
Since $1 \in S$, so, $m > 1$ and so $m-1$ is a natural number.
Again since m is the least element in T, $m-1$ is not in T and so, $m-1 \in S$.
Since $m-1 \in S$, by (ii) $(m-1)+1 = m \in S$, which is a contradiction.
Therefore, our assumption is wrong and T is empty and the theorem is proved.

2.2.2 Let $P(n)$ be a statement involving a natural number n. If (i) $P(1)$ is true, and

(ii) $P(k+1)$ is true whenever $P(k)$ is true, where k is a natural number, then

$P(n)$ is true for all natural numbers n.

Proof: Let $S = \{ n \in \mathbb{N} : P(n) \text{ is true} \}$

Then S has the properties:

(i) $1 \in S$

(ii) $k \in S \Rightarrow k+1 \in S$

So, by the principle of mathematical induction,

$S = \mathbb{N}$

So, $P(n)$ is true for all natural numbers n.

Note: 1. To establish a property involving natural numbers by the principle of Mathematical induction, both the conditions (i) and (ii) must be established.
The condition (i) is called the basis of Mathematical induction and the assumption made in the condition (ii) is called the induction hypothesis.

2. Principle of Mathematical induction is also written as principle of induction.

Worked Examples

1. Use the principle of mathematical induction to prove that

$1+3+5+\cdots+(2n-1) = n^2$ for all $n \in \mathbb{N}$.

Solution: Let $P(n)$ be the statement

$$1+3+5+\cdots+(2n-1) = n^2$$

Step 1. $P(1)$ is true as $1 = 1^2$

Step 2. Let $P(k)$ be true for some $k \in \mathbb{N}$

So, $1+3+5+\cdots+(2k-1) = k^2$

So, $1+3+5+\cdots+(2k-1)+(2k+1) = k^2+2k+1 = (k+1)^2$

So, $P(k+1)$ is true

So, By principle of mathematical induction, $P(n)$ is true for all natural numbers n.

So, $1+3+5+\cdots+(2n-1) = n^2$ for all $n \in \mathbb{N}$.

2. Prove that $3^{2n}-8n-1$ is divisible by 64 for all $n \in \mathbb{N}$

Proof: Let $P(n)$ be the statement

$f(n) = 3^{2n}-8n-1$ is divisible by 64

Step 1. $f(1) = 9-8-1 = 0$. So, $f(1)$ is divisible by 64.

So, $P(1)$ is true

Step 2.
$$\begin{aligned} f(k+1)-f(k) &= \left[3^{2k+2}-8(k+1)-1\right]-\left[3^{2k}-8k-1\right] \\ &= 8\left(3^{2k}-1\right) = 8\left(9^k-1\right) \\ &= 8\cdot 8\left(9^{k-1}+9^{k-2}+\cdots+1\right) \\ &= 64p \text{ where } p \text{ is an integer} \end{aligned}$$

So, $f(k+1)$ is divisible by 64 if $f(k)$ is so.

This proves that $P(k+1)$ is true whenever $P(k)$ is true.
So, by the principle of Mathematical induction, $P(n)$ is true for all natural numbers n.
So, $3^{2n}-8n-1$ is divisible by 64 for all $n \in \mathbb{N}$.

There is a variation of the principle of mathematical induction.
Let S be a non-empty subset of $\mathbb{N}$ such that

(i) $n_0 \in S$ and

(ii) $k(\geq n_0) \in S \Rightarrow k+1 \in S$

then $S = \{n \in \mathbb{N} : n \geq n_0\}$

We can utilize this principle to prove that if
$P(n)$ be a statement involving a natural number n satisfying the conditions:

(i) $P(n_0)$ is true for some natural number n_0

(ii) for $k \geq n_0$, $P(k)$ ~~⇒ P()~~ is true $\Rightarrow P(k+1)$ is true

then $P(n)$ is true for all $n \geq n_0$

Worked Example

3. Prove that $n! > 2^n$ for all natural numbers $n \geq 4$

Proof: Let $P(n)$ be the statement $n! > 2^n$
Here $P(1)$, $P(2)$, $P(3)$ are not true. But ~~...~~ $P(4)$ is true as $4! = 4\times3\times2\times1 = 24 > 2^4 = 16$
So, assume that $P(k)$ is true for $k \geq 4$
Then $k! \geq 2^k$
So, $(k+1)! > 2^k.(k+1) > 2^{k+1}$, since $k+1 > 2$
So, $P(k+1)$ is true when $P(k)$ is true for $k \geq 4$
So,