a set which is open (resp. closed) in Y is not necessarily open (resp. closed) in X.

1.11.3 Let Y be a subspace of the usual metric space $\mathbb{R}_u$

1. If $Y = [0,1]$, then the set $[0, \frac{1}{2})$ is open in Y but not in $\mathbb{R}_u$
2. If $Y = (0,1)$, then the set $(0, \frac{1}{2}]$ is closed in Y but not in $\mathbb{R}_u$

1.11.4 Lemma

Let (Y, d_Y) be a subspace of a metric space (X, d). If $a \in Y$ and $r > 0$, then $B'_r(a) = Y \cap B_r(a)$ where $B_r(a)$ and $B'_r(a)$ are open balls, respectively, in (X, d) and (Y, d_Y).

Proof : Exercise

1.11.5 Theorem

Let (Y, d_Y) be a subspace of a metric space (X, d) and $A \subset Y$. Then

(a) A is open in Y if and only if $\exists$ an open set G in X such $A = G \cap Y$

(b) A is closed in Y if and only if $\exists$ a closed set F in X such that $A = F \cap Y$.

Proof : (a) We will use the symbol $B_r(x)$ and $B'_r(x)$, for the open balls centred at x with radius $r > 0$, respectively, for the spaces (X, d) and (Y, d_Y).

Suppose first that $A = G \cap Y$. In order to prove that
A is open in Y, let $x \in A$ be arbitrary. Then $x \in G$
and $x \in Y$. Since G is open in X, $\exists$ an $r > 0$ such
that $B_r(x) \subset G$. Also since $x \in Y$, we have
~~So~~ $B'_r(x) = B_r(x) \cap Y \subset G \cap Y = A$. This verifies
that x is an interior point of A as a subset
of the metric space (Y, d_Y). Since $x \in A$ being
arbitrary it follows that $A^0 = A$ in (Y, d_Y)

Hence A is open in Y.

Conversely, let A be open in Y and let $x \in A$
be arbitrary. Then $\exists$ an open ball $B'_{r_x}(x)$ such
that $B'_{r_x}(x) \subset A$. Now

$$A = \bigcup_{x \in A} B'_{r_x}(x) = \bigcup_{x \in A} \left(B_r(x) \cap Y\right) = \left(\bigcup_{x \in A} B_r(x)\right) \cap Y$$

$$= G \cap Y \quad \text{where } G = \bigcup_{x \in A} B_{r_x}(x)$$

But G being an arbitrary union of open balls in X,
is an open set in X. Hence $A = G \cap Y$, where G
is open in X.

(b) We have

$$A \text{ is closed in } Y \Leftrightarrow Y - A \text{ is open in } Y$$

$$\Leftrightarrow Y - A = G \cap Y, \text{ (by part (a)) which } G \text{ is open in } X$$

$$\Leftrightarrow A = Y - G \cap Y$$

$$\Leftrightarrow A = X \cap Y - G \cap Y$$

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$\Leftrightarrow A = (X - G) \cap Y$

$\Leftrightarrow A = F \cap Y$ where $F = X - G$ is a closed set in X.

1.11.6 Corollary

Let (Y, d_Y) be a subspace of a metric space (X, d) and $A \subset Y$. Then

(a) A is open in Y and Y is open in $X \Rightarrow A$ is open in X.

(b) A is closed in Y and Y is closed in $X \Rightarrow A$ is closed in X.

1.11.7 Theorem

Let (Y, d_Y) be a subspace of a metric space (X, d) and $A \subset Y$. Then

(a) $x \in Y$ is a limit point of A in Y if and only if x is a limit point of A in X.

(b) The closure of A in Y is $\bar{A} \cap Y$, where $\bar{A}$ is closure of A in X.

Proof : (a) Let $x \in Y$ be a limit point of A in Y. Then $(B_r'(x) - \{x\}) \cap A \neq \phi$ ... (1) for each $r > 0$, where $B_r'(x)$ denotes an open ball in Y. Now for any given $r > 0$, we have

$$(B_r(x) - \{x\}) \cap A = (B_r(x) \cap Y - \{x\}) \cap A \quad (\because A \subset Y)$$
$$= (S_r'(x) - \{x\}) \cap A$$
$$\neq \phi \quad (\text{by } (1))$$

This proves that x is a limit point of A in X.

The converse can be established by retracing the above steps.

(b) Let B denotes the closure of A in Y. The set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed in Y (Theorem 1.11.5). Since $\bar{A} \cap Y$ contains A, and since by Theorem 1.8.2, B equals the intersection of all closed subsets of Y containing A, we must have $B \subset \bar{A} \cap Y$.

In order to have the reverse inclusion, note that B is closed in Y. As such, by Theorem 1.11.5, $B = F \cap Y$, for some set F closed in X. Then F is closed set of X containing A. Since $\bar{A}$ is the intersection of all such closed sets, we conclude that $\bar{A} \subset F$. Hence $\bar{A} \cap Y \subset F \cap Y = B$

2.1 Convergent Sequence

2.1.1 Definition: Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be convergent if $\exists$ a point $x \in X$ such for each $\varepsilon > 0$, $\exists$ a positive integer N such that $d(x_n, x) < \varepsilon$, $\forall\, n \geq N$

or equivalently, for each open ball $B_\varepsilon(x)$, centred on x, $\exists$ a positive integer N such that

$$x_n \in B_\varepsilon(x), \quad \forall\, n \geq N$$

We usually symbolize this by writing $x_n \to x$ or $\lim_{n\to\infty} x_n = x$