and we express it by saying that $x_n$ approaches $x$ or that $x_n$ converges to $x$. The element $x$ is called the limit of the sequence $\{x_n\}$

Note: A sequence which is not convergent is said to be divergent.

2.1.2 Theorem

In a metric space every convergent sequence has a unique limit.

Proof: Let $(X, d)$ be a metric space and $\{x_n\}$ be a ~~converg~~ convergent sequence in $X$. Let, if possible, the sequence $\{x_n\}$ converges to two points $x$ and $y$.

Then for each $\epsilon > 0$, $\exists$ positive integers $N_1$ and $N_2$ such that $d(x_n, x) < \frac{\epsilon}{2}$, $\forall n \geq N_1$

and $d(x_n, y) < \frac{\epsilon}{2}$, $\forall n \geq N_2$

Now $d(x, y) \leq d(x_n, x) + d(x_n, y)$ (by triangle inequality)

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq N = \max\{N_1, N_2\}$$

$$\Rightarrow \quad x = y$$

This verifies that the limit is unique.

2.1.3 Definition

In a metric space, a sequence is said to be bounded, if the range of the sequence forms a bounded set.

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P 2.1.4 Theorem

In a metric space, every convergent sequence is bounded.

Proof: Let $(X, d)$ be a metric space and $\{x_n\}$ be a convergent sequence in $X$ such that $x_n \to x$ as $n \to \infty$, in $(X, d)$. Then $\exists$ a positive integer $N$ such that

$$d(x_n, x) < 1 \quad \forall n \geq N$$

write $\quad r = \max\{1 ; d(x_n, x), 1 \leq n \leq N\}$

So, $\quad d(x_n, x) \leq r \;, \; \forall n \in \mathbb{N}$ (the set of all natural number)

so that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$$
$$\leq 2r \quad \forall n, m \in \mathbb{N}.$$

$\Rightarrow$ The diameter of the range of the sequence is bounded by $2r$.

This proves ~~the~~ the result.

Remark: The property of the convergence of a sequence in a metric space $(X, d)$ is not inherent in a sequence but depends on the space and the metric used.

2.1.5 Examples

1. Convergence depends on the space $X$. Let us consider the usual metric space $\mathbb{R}_u$. Let $\{x_n\}$ be a sequence in $\mathbb{R}_u$, where $x_n = \frac{1}{n}$. Note that $x_n \to 0 \in X$ as $n \to \infty$. However if we take $X = (0,1)$ with usual metric and take the same

sequence $\{x_n\}$, $x_n = \frac{1}{n}$ $(n \in \mathbb{N})$, we note that

$$x_n \to 0 \notin X$$

As such $\{x_n\}$ does not converge in this case.

2. Convergence depends on the metric used.

Consider the sequence $\{x_n\}$ in the space $C[0,1]$,

where $x_n(t) = e^{-nt}$, $n \in \mathbb{N}$

we find that $x_n \to 0$ with respect to the metric $d_1$ on $C[0,1]$ since

$$d_1(x_n, 0) = \int_0^1 e^{-nt}\,dt = \frac{1}{n}\left(1 - e^{-n}\right) \to 0 \text{ as } n \to \infty$$

On the other hand, the same sequence $\{x_n\}$ is not convergent with respect to the metric $d_\infty$ on $C[0,1]$ since

$$d_\infty(x_n, 0) = \max_{t \in [0,1]} \left|e^{-nt}\right| = 1, \ \forall n \in \mathbb{N}$$

$$\not\to 0 \text{ as } n \to \infty$$

Note: These facts can also be verified by considering the same sequence in $\mathbb{R}$, when $\mathbb{R}$ is equipped with the usual metric and it is equipped with discrete metric.

We give below a characterisation of limit points of a set (Definition 1.7.1) in terms of convergent sequences.

2.1.6 Theorem

Let $(X, d)$ be a metric space and $A \subset X$. Then $x \in X$

is a limit point of A if and only if $\exists$ a sequence $\{x_n\}$ of points in A, none of which equals $x$, such that $\lim_{n \to \infty} x_n = x$

Proof: Let $x \in X$ be a limit point of A. Construct a sequence $\{x_n\}$ by recursion as follows:

Take $x_1 \in (B_1(x) - \{x\}) \cap A$; this is possible, since, $x$ being a limit point of A, $(B_1(x) - \{x\}) \cap A \neq \phi$.

Likewise the points $x_1, x_2, \ldots, x_n$ ~~having chosen~~ have been chosen such that $x_i \in (B_{1/i}(x) - \{x\}) \cap A$, for $i = 1, 2, \ldots, n$. Still $(B_{1/n+1}(x) - \{x\}) \cap A$ being non-empty, it is possible to choose $x_{n+1}$ to be a point on this set and process is repeated infinitely many times. Thus the sequence $\{x_n\}$ has been constructed by recursion; all the points of which are in A and none of which equals $x$.

Now, let $\varepsilon > 0$ be given and let N be a positive integer such that $N > \frac{1}{\varepsilon}$. Then

$\cancel{x_n \in B_{1/n}(x) \subset B}$

$$d(x_n, x) < \frac{1}{n} < \frac{1}{N} < \varepsilon \quad \forall \; n > N$$

Hence $\lim_{n \to \infty} x_n = x$

Conversely, assume that there is a sequence $\{x_n\}$ of points in A, none of which equals $x$,

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