such that $\lim_{n\to\infty} x_n = x$. So, for a given $\varepsilon > 0$, $\exists$ a positive integer N such that

$$x_n \in B_\varepsilon(x), \quad \forall\, n > N$$

Then $\left(B_\varepsilon(x) - \{x\}\right) \cap A \neq \phi$. Hence x is a limit point of A.

2.1.7 Corollary

Let (X, d) be a metric space and $A \subset X$. Then A is closed if and only if every convergent sequence of points of A has its limit in A.

2.1.8 Theorem

Let $\{x_n\}$ be a sequence in a metric space (X, d) such that $\lim_{n\to\infty} x_n = x$. Let A be the range of the sequence $\{x_n\}$. Then

(a) If A is a finite set, then $x_n = x$ for infinitely many n

(b) If A is infinite set, then x is a limit point of A.

Proof: (a) straightforward

(b) Let, if possible, x be not a limit point of A. Then $\exists$ an open ball $B_\varepsilon(x)$ such that $\left(B_\varepsilon(x) - \{x\}\right) \cap A = \phi$

i.e. $B_\varepsilon(x)$ contains no point of A other than x. But x is the limit of the

~~sq~~ sequence $\{x_n\}$. So, for each $\varepsilon > 0$, $\exists$ a positive integer N such that

$$d(x_n, x) < \varepsilon \text{ or } x_n \in B_\varepsilon(x) \ \forall n > N$$

This is a contradiction. Hence the assertion (b) follows.

2.2 CAUCHY SEQUENCE

2.2.1 Definition

A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for each $\varepsilon > 0$, $\exists$ a positive integer N such that $d(x_m, x_n) < \varepsilon \quad \forall m, n \geq N$

2.2.2 Theorem

Every convergent sequence is a cauchy sequence

Proof: Let $x_n \to x$ as $n \to \infty$. Then, for each $\varepsilon > 0$, $\exists$ a positive integer N such that

$$d(x_n, x) < \frac{\varepsilon}{2}, \quad \forall n \geq N$$

Now, $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x)$ (triangle inequality)

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall m, n \geq N$$

So, $\{x_n\}$ is a Cauchy sequence.

Remark: In a metric space, every convergent sequence is a Cauchy sequence (Theorem 2.2.2). But the converse is not true

2.2.3 Example

1. Consider the sequence $\{x_n\}$ in the usual metric space $\mathbb{Q}_u$, where

$x_1 = \cdot 1$
$x_2 = \cdot 101$
$x_3 = \cdot 101001$
$x_4 = \cdot 1010010001$
... .
....

It is easy to verify (verify it!) that $\{x_n\}$ is a Cauchy sequence which does not converge in $\mathbb{Q}_u$

2. Let $X = (0, 1]$ be the metric space in the usual metric and $\{x_n\}$, where $x_n = 1/n$, be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence since for each $\varepsilon > 0$, we have

$$d(x_m, x_n) = \left|\frac{1}{m} - \frac{1}{n}\right| < \varepsilon, \ \forall m, n > \left[\frac{1}{\varepsilon}\right] + 1$$

where $\left[\frac{1}{\varepsilon}\right]$ = integral part of $\frac{1}{\varepsilon}$

On the other hand, $x_n \to 0 \notin X$.

Note that if in this example, if we take $X = [0, 1]$, then the sequence is Cauchy as well as convergent.

2.3. COMPLETE METRIC SPACES

2.3.1 Definition: A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

(In other words, (X, d) is a complete metric space, if, whenever the sequence $\{x_n\}$ in X is such that $d(x_m, x_n) \to 0$ as $m, n \to \infty$ $\exists$ an $x \in X$ with $d(x_n, x) \to 0$ as $n \to \infty$)

~~1. The usual metric space~~

2.3.2 Examples (Complete metric spaces)

1. The usual metric space $\mathbb{R}_u$ is complete (Prove it)

2. The Euclidean space $\mathbb{R}^n$ (Example 1.1.8 (8)) is a complete metric space

Proof: Let $\{x_m\}$ be a Cauchy sequence in $\mathbb{R}^n$ where $x_m = (\alpha_1^{(m)}, \alpha_2^{(m)}, \ldots, \alpha_n^{(m)})$

Then, for each $\varepsilon > 0$, $\exists$ a positive integer N such that $d(x_m, x_p) = \left(\sum_{i=1}^{n} (\alpha_i^{m} - \alpha_i^{(p)})^2 \right)^{1/2} < \varepsilon, \forall m, p > N$ ...(1)

On squaring both sides, we get

$$\sum_{i=1}^{n} \left(\alpha_i^{(m)} - \alpha_i^{(p)}\right)^2 < \varepsilon^2$$

$$\Rightarrow \quad \left(\alpha_i^{(m)} - \alpha_i^{(p)}\right)^2 < \varepsilon^2$$

$$\Rightarrow \quad \left|\alpha_i^{(m)} - \alpha_i^{(p)}\right| < \varepsilon \quad \forall m, p \geq N \quad (i = 1, 2, \ldots, n)$$

This shows that for each fixed i ($i = 1, 2, \ldots, n$) the sequence $\{\alpha_i^{(m)}\}_m$ is a Cauchy sequence in the usual metric space $\mathbb{R}$. Let $\alpha_i^{(m)} \to \alpha_i$ as $m \to \infty$ (as $\mathbb{R}$ is complete, $\{\alpha_i^{(m)}\}$ converges)

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