~~As~~, Using ~~the~~ these n limits, we define $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Clearly, $x \in \mathbb{R}^n$. Letting $p \to \infty$ in (1), we ~~obtain~~ obtain

$$d(x_m, x) \leq \varepsilon \quad \forall m \geq N$$

$$\Rightarrow \quad x_m \to x \quad \text{in } \mathbb{R}^n.$$

Hence $\mathbb{R}^n$ is a complete metric space.

2.3.3 Examples (Incomplete metric space)

1. The space ~~$\mathbb{Q}$~~ $\mathbb{Q}_u$ with the usual metric of absolute value is not complete (Example 2.2.3(1))

2. The metric space (X, d), where $X = (0, 1]$ and d is the usual metric on X, is not complete (Example 2.2.3 (2))

2.3.4 Theorem

Let (Y, d_Y) be a subspace of the metric space (X, d). Then Y is complete implies that Y is closed.

Proof: To prove that Y is closed, let x be a limit point of Y. Then every open ball centred on x contains points (other than x) of Y. In particular, the open ball $B_{1/n}(x)$, where n is a positive integer, contain a point x_n of Y, other than x.

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Thus $\{x_n\}$ is a sequence in Y such that $x_n \to x$ in X since $d(x_n, x) < \frac{1}{n}$. In view of Theorem 2.2.2, $\{x_n\}$ is a Cauchy sequence in X and hence in Y. But Y being complete, $x \in Y$. Hence Y is closed.

<u>2.3.5. Theorem</u>

Let (X, d) be a complete metric space and (Y, d_Y) be a subspace of (X, d). Then Y is closed implies Y is complete.

Proof: Let $\{x_n\}$ be a Cauchy sequence in Y. Then it is a also a Cauchy sequence in X since $Y \subset X$. But X being complete, $\{x_n\}$ converges to a point $x \in X$. We claim that $x \in Y$.

Now, there arise two cases:

<u>Case 1</u> If $\{x_n\}$ has only finitely many distinct points, then $x_n = x$ for infinitely many values of n. Since $\{x_n\}$ is in Y, it follows that $x \in Y$.

<u>Case 2</u> If $\{x_n\}$ consists of infinitely many distinct points, then the limit of the sequence is also the limit point of the range of the sequence $\{x_n\}$ (by Theorem 2.1.8). So, x is also a limit point of Y since $\{x_n\} \subset Y$. But Y being closed, $x \in Y$. This completes the proof of the theorem.

2.3.6 Corollary

Let (X, d) be a metric space and (Y, d_Y) be a subspace of (X, d). Then Y is complete if and only if Y is closed.

2.3.7 Theorem (Cantor's Intersection Theorem)

Let (X, d) be a complete metric space and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$. Then, the intersection $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Proof: Construct a sequence $\{x_n\}$ in X by selecting a point $x_n \in F_n$ for each n. Since the sets F_n are ~~closed~~ nested, $x_n \in F_m$, $\forall n \geq m$. We now prove that $\{x_n\}$ is a Cauchy sequence.

Let $\varepsilon > 0$ be given. Since $d(F_n) \to 0$, $\exists$ a positive integer N such that

$$d(F_N) < \varepsilon$$

Note that $x_n, x_m \in F_N$ $\forall n, m \geq N$ and as such, we have $d(x_n, x_m) \leq d(F_N) < \varepsilon$ $\forall n, m \geq N$

Thus $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, $\exists$ $x \in X$ such that $x_n \to x$

Now, we claim that $x \in \bigcap_{n=1}^{\infty} F_n$. Let n be fixed.

Then the subsequence $\{x_n, x_{n+1}, \dots\}$ of $\{x_n\}$ is contained in F_n and still converges to x. But F_n being closed subspace of the complete metric space (X, d), it is complete and so $x \in F_n$.

This is true for each $n \in \mathbb{N}$. Hence $x \in \bigcap_{n=1}^{\infty} F_n$

This verifies that $\bigcap_{n=1}^{\infty} F_n$ is non-empty

Finally, to establish that x is only point in the intersection $\bigcap_{n=1}^{\infty} F_n$, let $y \in \bigcap_{n=1}^{\infty} F_n$. Then x and y both are in F_n for each n.

So, $0 \leq d(x, y) \leq d(F_n) \to 0$ as $n \to \infty$

$\Rightarrow d(x, y) = 0$

$\Rightarrow x = y$

This completes the proof of the Theorem

Remark: The assertion of Theorem 2.3.7 may not be true if either of the condition:

(a) each F_n is closed

(b) $d(F_n) \to 0$ as $n \to \infty$ is dropped

2.3.8 Example

Consider the usual metric space $\mathbb{R}_u$, which, of course is complete.

(a) Take $F_n = [n, \infty)$. Note $\{F_n\}$ is a sequence of non-empty closed sets such that $d(F_n) \not\to 0$ as $n \to \infty$ and that $\bigcap_{n=1}^{\infty} F_n = \phi$

(b) Take $F_n = (0, \frac{1}{n}]$. Note that $\{F_n\}$ is a