decreasing sequence of non-empty sets which are not closed, $d(F_n) \to 0$ as $n \to \infty$ and $\bigcap_{n=1}^{\infty} F_n = \phi$

1.3.9 Theorem

If in a metric space (X, d) every decreasing sequence $\{F_n\}$ of non-empty closed sets with $d(F_n) \to 0$ as $n \to \infty$ has exactly one point in its intersection, then (X, d) is complete.

Proof: Let $\{x_n\}$ be a Cauchy sequence in X.

Let $G_n = \{x_n, x_{n+1}, \dots\}$.

Then it is easy to verify that $d(G_n) \to 0$ as $n \to \infty$; Since $\{x_n\}$ is Cauchy sequence. This immediately leads to $d(\bar{G}_n) \to 0$ as $n \to \infty$ (Theorem 1.10.2)

Taking $F_n = \bar{G}_n$, we note $\{F_n\}$ is a decreasing sequence of non-empty ~~not~~ closed sets with $d(F_n) \to 0$ as $n \to \infty$. Then, by the hypothesis, $\exists$ an $x \in X$, such that $x \in \bigcap_{n=1}^{\infty} F_n$. So,

$$d(x, x_n) \le d(F_n) \quad \forall n$$
$$\to 0 \quad \text{as } n \to \infty$$

Hence $x_n \to x$ in X. This proves that X is complete.

3 CONTINUOUS ~~FUNCT~~ FUNCTIONS

3.1 DEFINITION AND CHARACTERISATIONS

3.1.1 Definition

Let (X, d_1) and ~~(X)~~ (Y, d_2) be two metric spaces.

A function $f: X \to Y$ is said to be continuous at a point $x_0 \in X$ if for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon$, $x \in X$ ...(1)

that is $x \in B_\delta(x_0) \Rightarrow f(x) \in B_\varepsilon(f(x_0))$ ...(2)

Which means the same thing as

$$f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)) \quad \ldots (3)$$

The function f is said to be continuous on X or simply continuous if it is continuous at each point of X.

3.1.2 Examples

1. In a metric space (X, d), the identity function is continuous
2. Let $\mathbb{R}_u$ be the usual metric space and $f: \mathbb{R} \to \mathbb{R}$ be a constant function. Then f is continuous.
3. Let $f: [0,1] \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

Then f is continuous ~~at~~ in $[0,1]$ except at $x = 1$

4. Let $f: [0,1] \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$$

Then f is continuous ~~at~~ only at $x = \frac{1}{2}$ in $[0,1]$

5. Let $f: \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

Then f is discontinuous at every point of $\mathbb{R}$.

(This function is called Dirichlet function)

3.1.3 Theorem

Let (X, d_1) and (Y, d_2) be metric spaces and $f: X \to Y$ be a function. Then, f is continuous at a point $x_0 \in X$ if and only if $f(x_n) \to f(x_0)$, for every sequence $\{x_n\} \subset X$ with $x_n \to x_0$

Proof: we first assume that f is continuous at x_0. Let $\varepsilon > 0$ be given and let $\{x_n\}$ be a sequence in ~~in~~ X such that $x_n \to x_0$,

Then $\exists$ a $\delta > 0$ such that $f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0))$

Also since $x_n \to x_0$, $\exists$ a positive integer N such that $x_n \in S_\delta(x_0)$, $\forall n \geq N$. Hence

$$f(x_n) \in S_\varepsilon(f(x_0)), \quad \forall n \geq N$$

This proves that $f(x_n) \to f(x_0)$

Conversely, let $f(x_n) \to f(x_0)$, for every sequence $\{x_n\}$ with $x_n \to x_0$. Let, if possible, f be not continuous at x_0. Then $\exists$ an $\varepsilon > 0$ such that there is no open ball centred at x_0 whose f-image is contained in $B_\epsilon(f(x_0))$. Consider the sequence of open balls

$$B_1(x_0), B_{1/2}(x_0), \ldots B_{1/n}(x_0) \ldots$$

Let $x_n \in B_{1/n}(x_0)$ be such that $f(x_n) \notin B_\epsilon(f(x_0))$.

Thus we get a sequence $\{x_n\} \subset X$ with $x_n \to x_0$ such that $f(x_n) \not\to f(x_0)$. This is a contradiction to the hypothesis. Hence f is continuous at x_0.

As a consequence of Theorem 4.1.3, we have

3.1.4 Theorem
Let (X, d_1) and (Y, d_2) be metric spaces and $f: X \to Y$ be a function. Then f is continuous if and only if for each $x \in X$, $f(x_n) \to f(x)$ for every sequence $\{x_n\} \subset X$ with $x_n \to x$
(In other words, f is continuous if and only if f preserves convergent sequences)

Remark The converse of Theorem 3.1.4 is not true. More precisely, for a continuous function $f: X \to Y$ if $f(x_n) \to f(x)$ then the assertion $x_n \to x$ is not necessarily true.
Indeed, consider the function $f: \mathbb{R}_u \to \mathbb{R}_u$ given by $f(x) = x^2$ and $x_n = (-1)^n$, $n \in \mathbb{N}$. Then $f(x_n) = 1 \to f(1)$ as $n \to \infty$

But $x_n \not\to 1$