Note: The result in Theorem 3.1.4 usually helps for proving, in a simple way, a function to be discontinuous at a point x by selecting a suitable sequence $\{x_n\}$ for which $x_n \to x$ while $f(x_n) \not\to f(x)$.

Remark: In view of theorem 3.1.4 the convergent sequences are preserved under continuous mappings. However, it is not necessarily true for Cauchy sequences. It may happen that $\{x_n\}$ is a Cauchy sequence in (X, d_1), $f: X \to Y$ is continuous and $\{f(x_n)\}$ is not a Cauchy sequence in (Y, d_2).

Consider for instance, the function $f: (0,\infty) \to (0,\infty)$ given by $f(x) = \frac{1}{x}$. Here $\{\frac{1}{n}\}$ is a Cauchy sequence in $(0,\infty)$ but $\{f(\frac{1}{n})\} = \{n\}$ is not a Cauchy sequence in $(0,\infty)$

Theorem 3.1.5 Let (X, d_1) and (Y, d_2) be metric spaces and $f: X \to Y$ be a function. Then f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y.

Proof: Let f be continuous and G be open in Y. We shall prove that $f^{-1}(G)$ is open in X. Take some $x \in f^{-1}(G)$. Then $f(x) \in G$. Since G is open, it follows that $B_\varepsilon(f(x)) \subset G$, for some $\varepsilon > 0$

But f being continuous, $\exists$ a $\delta > 0$ such that

$$f\left(B_\delta(x)\right) \subset B_\varepsilon(f(x)) \subset G$$

$$\Rightarrow \quad B_\delta(x) \subset f^{-1}(G)$$

Hence $f^{-1}(G)$ is open.

Conversely, assume that $f^{-1}(G)$ is open in X whenever G is open in Y. Let $x \in X$ be arbitrary and $\varepsilon > 0$ be given. Then $f(x) \in Y$ and $B_\varepsilon(f(x))$ is an open set. Therefore, by the assumption $f^{-1}\left(B_\varepsilon(f(x))\right)$ is an open set and $x \in f^{-1}\left(B_\varepsilon(f(x))\right)$. Consequently $\exists$ a $\delta > 0$ such that $B_\delta(x) \subset f^{-1}\left(B_\varepsilon(f(x))\right)$

i.e., $f\left(B_\delta(x)\right) \subset B_\varepsilon(f(x))$

So, f is continuous at x. Hence f is continuous as x is arbitrarily chosen.

Note: Theorem 3.1.5 provides a very general characterisation of continuous functions which is often used as a definition of continuity in the setting of topological spaces.

3.1.6 Theorem

Let (X, d_1), (Y, d_2) be metric spaces and $f: X \to Y$ be a function. Then, f is continuous if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y.

Proof: Let f be continuous and F be closed in Y. Then $Y-F$ is open in Y and therefore $f^{-1}(Y-F)$ is open in X (Theorem 3.1.5). Since $f^{-1}(F)$ is complement of $f^{-1}(Y-F)$, and $f^{-1}(Y-F)$ is open, it follows that $f^{-1}(F)$ is closed in X.

Conversely, suppose that $f^{-1}(F)$ is closed in X whenever F is closed in Y. We shall prove that f is continuous. Let G be an open subset of Y. Then $Y-G$ is closed in Y and by hypothesis $f^{-1}(Y-G)$ is closed in X. Using argument as in the first part, we note that $f^{-1}(G)$ is open in X.

Hence f is continuous

3.1.7 <u>Theorem</u> Let (X, d_1), (Y, d_2) be metric spaces and $f: X \to Y$ be a function. Then f is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$, for every subset A of X

Proof: Let f be continuous. $f^{-1}\left(\overline{f(A)}\right)$ is closed in X since $\overline{f(A)}$ is closed in Y (Theorem 3.1.6)

Now we have $f(A) \subset \overline{f(A)}$

$$\Rightarrow \quad A \subset f^{-1}\left(\overline{f(A)}\right)$$

$$\Rightarrow \quad \bar{A} \subset \overline{f^{-1}\left(\overline{f(A)}\right)}$$

$\Rightarrow \bar{A} \subset f^{-1}(\overline{f(A)})$ ($\because f^{-1}(\overline{f(A)})$ is closed)

$\Rightarrow f(\bar{A}) \subset \overline{f(A)}$

Conversely, let $f(\bar{A}) \subset \overline{f(A)}$, for every subset A of X. We shall prove that f is continuous.

Let F be any closed set in Y. Then $\bar{F} = F$

Now $f^{-1}(F)$ is a subset of X and so, by hypothesis

$$f\left(\overline{f^{-1}(F)}\right) \subset \overline{f(f^{-1}(F))}$$

$$\Rightarrow f\left(\overline{f^{-1}(F)}\right) \subset \bar{F} = F$$

$$\Rightarrow \overline{f^{-1}(F)} \subset f^{-1}(F)$$

But $f^{-1}(F) \subset \overline{f^{-1}(F)}$. As such

$f^{-1}(F) = \overline{f^{-1}(F)}$. Hence $f^{-1}(F)$ is closed in X. So f is continuous.

3.1.8. Theorem

Let (X, d_1), (Y, d_2) be metric spaces and $f: X \to Y$ be a function. Then, f is continuous if and only if $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$, for every subset B of Y.

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