Proof: Let f be continuous. Let $A = f^{-1}(B)$

Then $f(A) \subset B$

$\Rightarrow \overline{f(A)} \subset \overline{B}$

$\Rightarrow f(\overline{A}) \subset \overline{B}$ $\left(\because f(\overline{A}) \subset \overline{f(A)} \text{ by Theorem 3.1.7}\right)$

$\Rightarrow \overline{A} \subset f^{-1}(\overline{B})$

$\Rightarrow \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$

Conversely, let $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every subset B of Y.

We shall prove that f is continuous.

Let F be any closed set in Y. Then $\overline{F} = F$

Now by the hypothesis, we have

$$\overline{f^{-1}(F)} \subset f^{-1}(\overline{F}) = f^{-1}(F) \quad (\because \overline{F} = F)$$

but $f^{-1}(F) \subset \overline{f^{-1}(F)}$. As such $\overline{f^{-1}(F)} = f^{-1}(F)$.

Hence $f^{-1}(F)$ is closed in X. So, f is continuous.

As a consequence of Theorem 3.1.4, 3.1.5, 3.1.6, 3.1.7 and 3.1.8, we have the theorem

3.1.9 Theorem

Let (X, d_1) and (Y, d_2) be ~~metr~~ metric spaces and $f: X \to Y$ be a function. ~~Then f is~~

Then the following statements are equivalent:

(a) f is continuous

(b) for each $x \in X$, $f(x_n) \to f(x)$ for every sequence $\{x_n\} \subset X$ with $x_n \to x$

(c) $f^{-1}(G)$ is open in X whenever G is open in Y

(d) $f^{-1}(F)$ is closed in X whenever F is closed in Y

(e) $f(\bar{A}) \subset \overline{f(A)} \quad \forall A \subset X$

(f) $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}) \quad \forall B \subset Y$

3.1.10 Theorem

Let (X, d_1), (Y, d_2) and (Z, d_3) be three metric spaces. Suppose $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Then $g \circ f$, the composite of f and g, is continuous.

Proof: We know that $g \circ f : X \to Z$. Let G be an open set in Z. Then

$g^{-1}(G)$ is open Y ($\because g$ is continuous)

$\Rightarrow f^{-1}(g^{-1}(G))$ is open in X ($\because f$ is continuous) $\Rightarrow (f^{-1} \circ g^{-1})(G)$ is open in X

$\Rightarrow (g \circ f)^{-1}(G)$ is open in X ($\because f^{-1} \circ g^{-1} = (g \circ f)^{-1}$)

$\Rightarrow g \circ f$ is continuous.

3.2 Uniform Continuity.

3.2.1 Let (X, d_1) and (Y, d_2) be two metric spaces

A function $f: X \to Y$ is said to be uniformly continuous if for given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $d_1(x, x') < \delta$, $x, x' \in X \Rightarrow d_2(f(x), f(x')) < \varepsilon$

Remark: Uniform continuity implies continuity. But the converse need not be true. The justification of the statement is obvious from the definition of uniform continuity. For the converse which is not true, in general, we give a counter example.

3.2.2 Example

Let $\mathbb{R}_u$ be the usual metric space. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. It is easy to verify that f is continuous. We shall prove that f is not uniformly continuous. We prove this by showing that $\exists$ an $\varepsilon > 0$ for which no δ works. Take $\varepsilon = 1$. Let $\delta > 0$ be given. Let

$$x_1 = \frac{\delta}{2} + \frac{1}{\delta}, \quad x_2 = \frac{1}{\delta}$$

Then $|x_1 - x_2| = \frac{\delta}{2} < \delta$

But $|f(x_1) - f(x_2)| = \left| \left(\frac{\delta}{2} + \frac{1}{\delta}\right)^2 - \frac{1}{\delta^2} \right| = \frac{\delta^2}{4} + 1 > 1$

Thus, whatever δ may be, $\exists$ $x_1, x_2 \in \mathbb{R}$ such that $|x_1 - x_1| < \delta$, but $|f(x_1) - f(x_2)| > 1$

3.2.3 Theorem

Let (X, d) be a metric space and $A \subset X$. Then the

the function $f : X \to \mathbb{R}$ given by

$$f(x) = d(x, A), \quad x \in X \text{ is uniformly continuous.}$$

Proof: By triangle inequality,

$$d(x, a) \leq d(x, y) + d(y, a), \quad \forall a \in A, \; x, y \in X$$

On taking infimum, we get

$$\inf_{a \in A} d(x, a) \leq d(x, y) + \inf_{a \in A} d(y, a)$$

($\because d(x,y)$ is independent of a)

$$\Rightarrow \quad d(x, A) \leq d(x, y) + d(y, A)$$

$$\Rightarrow \quad d(x, A) - d(y, A) \leq d(x, y)$$

This is true for all $x, y \in X$. Therefore on, ~~interchanging~~ interchanging x and y, we $d(y, A) - d(x, A) \leq d(x, y)$

Thus $\quad |d(x, A) - d(y, A)| \leq d(x, y)$

So, for a given $\varepsilon > 0$, choosing a δ, such that $0 < \delta \leq \varepsilon$,

we have,

$$|f(x) - f(y)| = |d(x, A) - d(y, A)| \leq d(x, y) < \delta \leq \varepsilon$$

i.e $\quad |f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$

Hence f is uniformly continuous.

3.2.4 Theorem.

Composition of two uniformly continuous functions is a uniformly continuous function.

Proof: Exercise

3.2.5 Theorem

Let (X, d_1) and (Y, d_2) be metric space and