$f: X \to Y$ be a uniformly continuous function. If $\{x_n\}$ be a Cauchy sequence in $X$, then $\{f(x_n)\}$ is a Cauchy sequence in $Y$.

Proof: Since $f$ is uniformly continuous, for a given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$d_1(x, x') < \delta,\ x, x' \in X \Rightarrow d_2\left(f(x), f(x')\right) < \varepsilon$$

In particular, we have

$$d_1(x_n, x_m) < \delta \Rightarrow d_2\left(f(x_n), f(x_m)\right) < \varepsilon \quad \text{---(1)}$$

But $\{x_n\}$ is a Cauchy sequence in $X$, given a $\delta > 0$, $\exists$ a positive integer $N$ such that

$$d_1(x_n, x_m) < \delta \quad \forall n, m \geq N \quad \text{---(2)}$$

So, from (1) and (2) it follows that

$$d_2\left(f(x_n), f(x_m)\right) < \varepsilon, \quad \forall n, m \geq N$$

Hence $\{f(x_n)\}$ is a Cauchy sequence in $Y$.

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## 5. Compactness

### 5.1 Compact spaces and sets

#### 5.1.1. Definition

Let $(X, d)$ be a metric space.

(a) A collection $\mathscr{C} = \{G_\alpha : \alpha \in I\}$ of subsets of $X$ is said to be a cover of $X$ if $\bigcup_{\alpha \in I} G_\alpha = X$

(b) A subclass $\mathscr{C}'$ of a cover $\mathscr{C}$ is said to be a subcover of $\mathscr{C}$ if $\mathscr{C}'$ itself covers $X$.

(c) A subcover $\mathcal{C}'$ of $\mathcal{C}$ is said to be a finite subcover if $\mathcal{C}'$ has only a finite number of members.

(d) A cover $\mathcal{C}$ is said to be an open cover of X if every set of in $\mathcal{C}$ is an open set.

5.1.2. Examples

1. Let $\mathbb{R}_u$ be the usual metric space. Let

$$\mathcal{C} = \{(-n, n) : n \in \mathbb{N}\}$$

and. $\mathcal{C}' = \{(-2n, 2n) : n \in \mathbb{N}\}$

Then $\mathcal{C}$ is an open cover of $\mathbb{R}$ and $\mathcal{C}'$ is a subcover of $\mathcal{C}$

2. Let $(0,1)$ be the metric space with the usual metric. Let $\mathcal{C} = \left\{\left(0, 1-\frac{1}{n+1}\right) : n \in \mathbb{N}\right\}$

and $\mathcal{C}' = \left\{\left(0, 1-\frac{1}{4(n+1)}\right) : n \in \mathbb{N}\right\}$

Then $\mathcal{C}$ is an open cover of $(0,1)$ and $\mathcal{C}'$ is a subcover of $\mathcal{C}$

5.1.3. Definitions

Let $(X, d)$ be a metric space

(a) The space $(X, d)$ is said to be compact if every open cover of X has a finite subcover.

(b) A subspace $(Y, d_Y)$ of $(X, d)$ is said be compact if it is compact as a metric space in its own right.

(c) A subset $Y \subset X$ is said to be compact if it compact as a metric subspace.

5.1.4 Examples

1. The usual metric space $\mathbb{R}_u$ is not compact.

Indeed, if $\mathscr{C} = \{(-n, n) : n \in \mathbb{N}\}$ then $\mathscr{C}$ is an open cover of $\mathbb{R}$ since

(i) each set in $\mathscr{C}$ is an open set

(ii) if $x \in \mathbb{R}$, then $\exists$ positive integer $n_x > |x|$.

As such ~~$n_x, n_x$~~ $x \in (-n_x, n_x)$ which gives $\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n)$

Let $\{(-n_p, n_p) : 1 \le p \le k\}$ be any finite subcollection of $\mathscr{C}$ and let

$$n^* = \max\{n_1, n_2, \ldots, n_k\}.$$

Then $n^* \notin \bigcup_{i=1}^{k} (-n_i, n_i)$

Thus no finite subcollection of $\mathscr{C}$ covers $\mathbb{R}$.

Hence $\mathbb{R}$ is not compact.

2. A finite set in any metric space $(X, d)$ is compact. In particular, a finite metric space is compact.

3. The discrete metric space $X_d$, where $X$ is an infinite set, is not compact. Indeed, the family $\mathscr{C} = \{\{x\} : x \in X\}$ of subsets of $X$ is an open cover of $X$. But there is no finite subfamily of $\mathscr{C}$ which covers $X$. Further, any

infinite subset of X is not compact set in $X_d$.
In fact, a subset of X is compact if and only if it is finite (Prove it)

4. The set of positive real numbers $\mathbb{R}^+$ is not compact in the usual metric space $\mathbb{R}_u$.

5.1.4 Theorem

Let $(X, d)$ be a metric space. Then, a closed subset of X is compact.

Proof: Let Y be a non-empty closed subset of X. We are to show that $(Y, d_Y)$ is compact.
Let $\{G_\alpha : \alpha \in I\}$ be an open cover of Y. We may write $G_\alpha = H_\alpha \cap Y$ where $H_\alpha$ is open in X for each $\alpha \in I$. (Theorem 1.11.5).

Then $Y = \bigcup_{\alpha \in I} G_\alpha \subset \bigcup_{\alpha \in I} H_\alpha$

Now $X = Y \cup Y^c \subset \left[\bigcup_{\alpha \in I} H_\alpha\right] \cup Y^c$

$\Rightarrow X = \left[\bigcup_{\alpha \in I} H_\alpha\right] \cup Y^c$

Since Y is closed, $Y^c$ is open and so $\{H_\alpha : \alpha \in I\} \cup \{Y^c\}$ is an open cover of X.
But X being compact, every open cover of X admits a finite subcover. In particular, $\{H_\alpha : \alpha \in I\} \cup \{Y^c\}$ has a finite subcover.
In case, this finite subcover contains $Y^c$ as one of its members, remove it since it covers no part of Y. Let the remaining